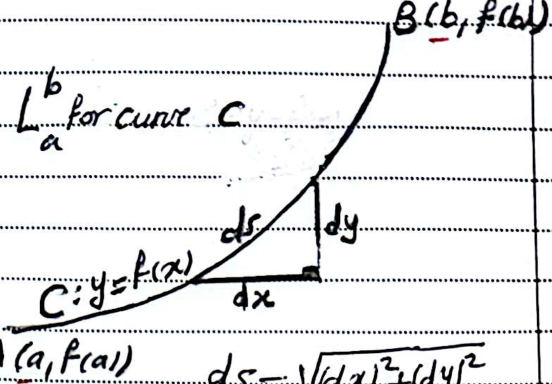
lecture (28) Arc length and surfaces of Revolution



*** Definition**

If $y = f(x)$ then the arc length L_a^b for curve C is defined as

$$L_a^b = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$

Similarly if $x = g(y)$ then

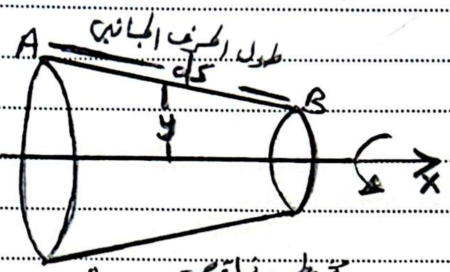
$$L_c^d = \int_c^d \sqrt{1+[g'(y)]^2}\, dy$$

$$ds = \sqrt{(dx)^2+(dy)^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$ds = \sqrt{1+[f'(x)]^2}\, dx$$

* The frustum of a cone generated by rotating the slanted line segment $\overline{AB}$ of length ds about the x-axis has area $2\pi y\, ds$ where $y = f(x)$ its average radius.



مخروط ناقص

frustum of a cone

*** Definition**

The surface area of the solid generated by revolving the curve $y = f(x)$ about x-axis is

$$S.A = 2\pi \int_a^b y\, ds$$

i.e. $$S.A = 2\pi \int_a^b y \sqrt{1+[f'(x)]^2}\, dx = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\, dx$$

Similarly

$$S.A = 2\pi \int_c^d g(y)\sqrt{1+[g'(y)]^2}\, dy$$

by revolving the curve $x = g(y)$ about y-axis.

EX ① If $f(x) = 3x^{2/3} - 10$, find the arc length of the graph of f from the point $A(8, 2)$ to $B(27, 17)$.

Ans:

$$L_a^b = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$f(x) = 3x^{2/3} - 10 \Rightarrow f'(x) = 2x^{-1/3} = \frac{2}{x^{1/3}}$$

$$L_8^{27} = \int_8^{27} \sqrt{1 + \frac{4}{x^{2/3}}}\, dx = \int_8^{27} \sqrt{\frac{x^{2/3} + 4}{x^{2/3}}}\, dx$$

$$L_8^{27} = \int_8^{27} \sqrt{x^{2/3} + 4}\cdot x^{-1/3}\, dx = \frac{3}{2}\int_8^{27} \sqrt{x^{2/3} + 4}\cdot \frac{2}{3} x^{-1/3}\, dx$$

$$\frac{d}{dx} x^{2/3} = \frac{2}{3} x^{-1/3}$$

$$L_8^{27} = \cancel{\frac{3}{2}} \left[\frac{(x^{2/3} + 4)^{3/2}}{\cancel{3/2}}\right]_8^{27} = (13)^{3/2} - (8)^{3/2} \simeq 24.2$$

EX ② The graph of $y = \sqrt{x}$ from $(1,1)$ to $(4,2)$ is revolved about x-axis. Find the area of the resulting surface.

Ans:

$$S.A = 2\pi \int_a^b y\, ds$$

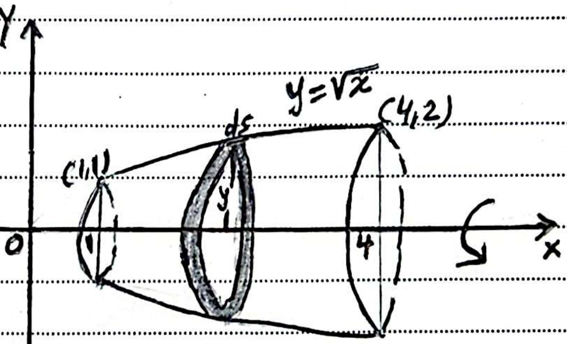
$$= 2\pi \int_1^4 f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$$= 2\pi \int_1^4 \sqrt{x} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$= \cancel{2}\pi \int_1^4 \cancel{\sqrt{x}}\, \frac{\sqrt{1 + 4x}}{\cancel{2}\cancel{\sqrt{x}}}\, dx$$

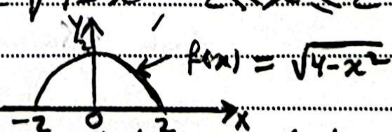
$$= \frac{\pi}{4} \int_1^4 (1 + 4x)^{1/2}\, 4\, dx = \frac{\pi}{4}\left[(4x+1)^{3/2} \cdot \frac{2}{3}\right]_1^4$$

$$\therefore S.A = \frac{\pi}{6}\left[17^{3/2} - 5^{3/2}\right] \simeq 30.85 \text{ sq. unit.}$$



HW EX ③ Find the length of the curve $f(x) = \sqrt{4 - x^2}$, $-2 \le x \le 2$

Ans: 2π



$$\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

HW EX ④ Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \le x \le 2$, about x-axis.

Ans: $\frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2})$.