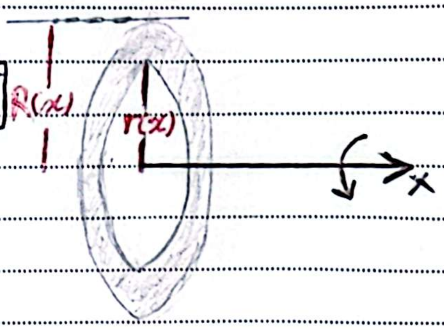
Lecture 26 II b. Volume Using Cross-Sections

[2] Solids of Revolution: the Washer Method

The area of washer is

$$A(x) = \pi\left[(\text{outer radius})^2 - (\text{inner radius})^2\right]$$

i.e. $A(x) = \pi([R(x)]^2 - [r(x)]^2)$



Volume by washers for rotation about x-axis is

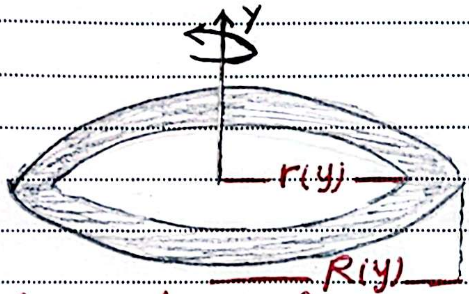
$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx$$

Similarly, volume by washers for rotation about y-axis is

$$V = \int_c^d A(y)\,dy = \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy$$

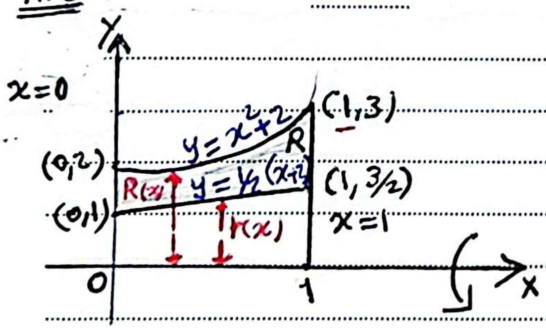
where washer's area is

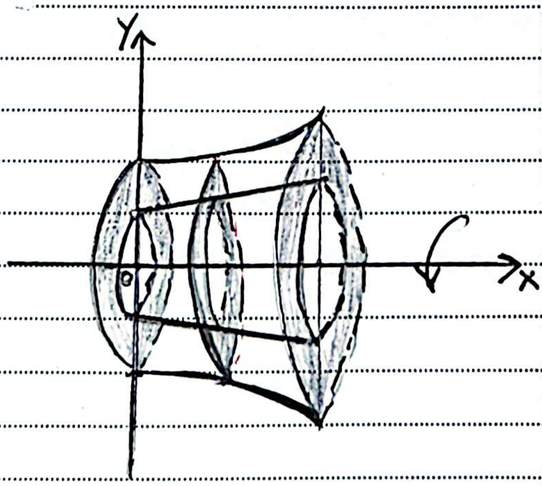
$A(y) = \pi([R(y)]^2 - [r(y)]^2)$



Ex ① The region bounded by the graphs of the equations $x^2 = y - 2$ and $2y - x - 2 = 0$ and the vertical lines $x = 0$ and $x = 1$. Find the volume of the resulting solid by revolving about x-axis.

• Ans





Note that,
the outer radius is $R(x) = x^2 + 2$, and
the inner radius is $r(x) = \frac{1}{2}(x + 2)$

The volume of solid by revolving about x-axis is

$$V = \pi \int_0^1 \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$= \pi \int_0^1 \left[(x^2+2)^2 - \frac{1}{4}(x+2)^2 \right] dx$$

$$= \pi \int_0^1 \left(x^4 + \frac{15}{4}x^2 - x + 3 \right) dx$$

$$= \pi \left[\frac{x^5}{5} + \frac{15}{12}x^3 - \frac{x^2}{2} + 3x \right]_0^1$$

$$(x^2+2)^2 - \frac{1}{4}(x+2)^2$$
$$= x^4 + 4x^2 + 4 - \frac{1}{4}x^2 - x - 1$$
$$= x^4 + \frac{15}{4}x^2 - x + 3$$

$$\therefore V = \pi \left(\frac{1}{5} + \frac{\cancel{15}^{5}}{\cancel{12}_{4}} - \frac{1}{2} + 3 \right) = \frac{79}{20}\pi \quad \#$$

EX ② Sketch the region R bounded by the graphs of the equations $y^2 = x$, $2y = x$, and find the volume of solid by revolving R about Y-axis.

Ans:

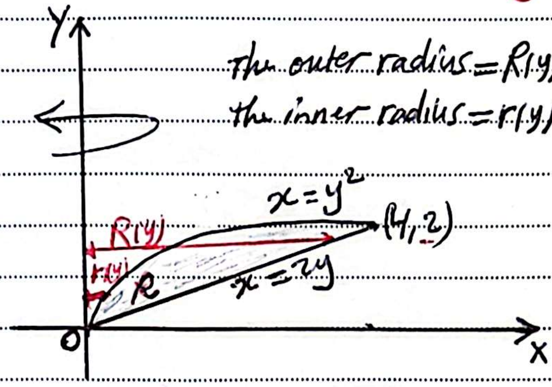
$y^2 = x$ ①, $2y = x$ ②

Solving ①, ② $\Rightarrow y^2 = 2y$

$\Rightarrow y(y-2) = 0$ i.e. $y = 0$, $y = 2$

① $\Rightarrow x = 0$, $x = 4$.

∴ the intersection points are $(0,0)$ and $(4,2)$ as shown in Fig.



The outer radius $= R(y) = 2y$,
the inner radius $= r(y) = y^2$

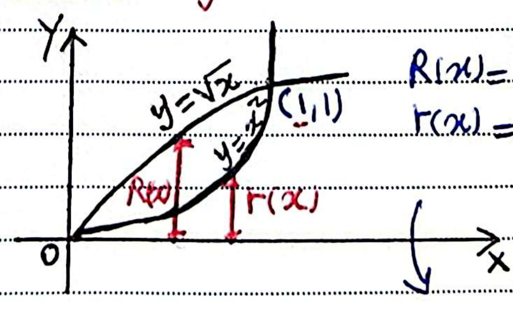
The volume of solid by revolving R about Y-axis is

$$V = \pi \int_0^2 \left([R(y)]^2 - [r(y)]^2 \right) dy$$

$$V = \pi \int_0^2 \left[(2y)^2 - (y^2)^2 \right] dy, \therefore V = \pi \left[\frac{4y^3}{3} - \frac{y^5}{5} \right]_0^2 = \frac{64}{15}\pi$$

HW EX ③ The region bounded by the curves $y = x^2$ and $y = \sqrt{x}$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Ans:



$R(x) = \sqrt{x}$,
$r(x) = x^2$

$$V = \frac{3\pi}{10} \quad \#$$