Lecture (25) (II)a. Volume Using Cross-Sections

Definition (1)

The volume of a solid of integrable cross sectional area $A(x)$ from $x=a$ to $x=b$ is the integral of A from a to b,

$$V=\int_a^b A(x)\,dx \qquad (1)$$

Definition (2)

The solid generated by rotating (or revolving) a planar region about an axis in its plane is called a solid of revolution.

[I] Solids of Revolution: The Disk Method

∵ The area of a circular disk of radius $r(x)$ is

$$A(x)=\pi(\text{radius})^2=\pi[r(x)]^2$$

∴ The volume of a solid by revolving about the x-Axis is

$$V=\int_a^b A(x)\,dx=\int_a^b \pi[r(x)]^2\,dx \qquad (2)$$

Similarly, the area of a circular disk of radius $r(y)$ is

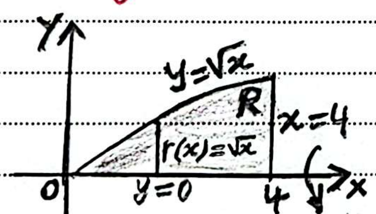
$$A(y)=\pi(\text{radius})^2=\pi[r(y)]^2$$

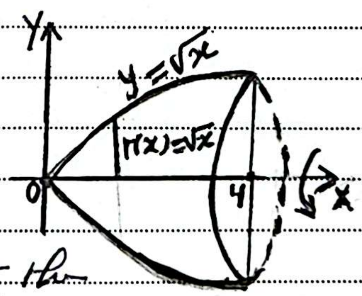
and the volume of a solid by revolving about Y-Axis is

$$V=\int_c^d A(y)\,dy=\int_c^d \pi[r(y)]^2\,dy \qquad (3)$$

Ex (1), Sketch the region R bounded by the graph of the equations $y=\sqrt{x}$, $x=4$, $y=0$ and find the volume of the solid generated if R is revolved about the x-axis.

Ans:





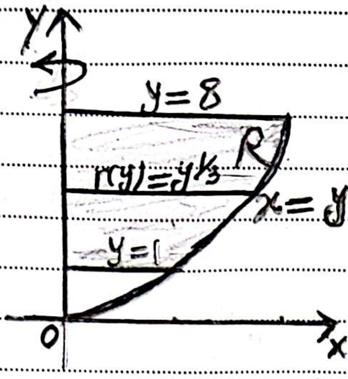
The volume of solid by revolving about the x-axis is

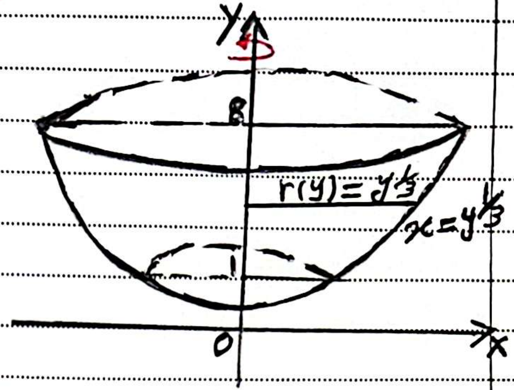
$$(2) \Rightarrow V=\pi\int_0^4[r(x)]^2\,dx=\pi\int_0^4 x\,dx$$

$$=\pi\left[\frac{x^2}{2}\right]_0^4=8\pi$$

EX (2) The region bounded by the y-axis and the graph of $y = x^3$, $y = 1$ and $y = 8$ is revolved about the y-axis. Find the volume of the resulting solid.

Ans:





The volume of solid by revolving about Y-axis is

(3) $$V = \pi \int_1^8 [r(y)]^2 dy = \pi \int_1^8 (y^{\frac{1}{3}})^2 dy$$

$$= \pi \int_1^8 y^{2/3} dy = \pi \left[\frac{y^{5/3}}{5/3} \right]_1^8$$

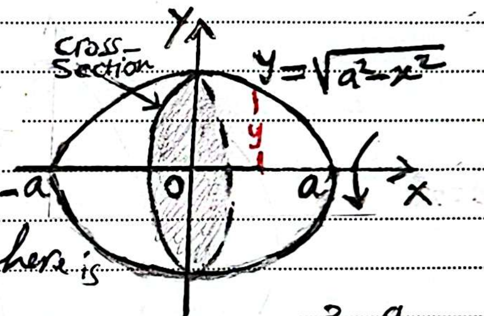
$$= \frac{3\pi}{5}[32 - 1] = \frac{93}{5}\pi \approx 58.4 \quad \#$$

$\frac{3}{5}\pi(2^5 - 1)$

$\frac{3(31)}{5}\pi$ ✓

EX (3) The circle $x^2 + y^2 = a^2$ is rotated about the x-axis to generate a sphere. Find the volume of the sphere.

Ans: ∵ the cross-sectional area is



$$A(x) = \pi [r(x)]^2 = \pi y^2 = \pi(a^2 - x^2)$$

∴ The volume of the resulting sphere is

$$V = \int_{-a}^{a} A(x) dx = \int_{-a}^{a} \underbrace{\pi(a^2 - x^2)}_{\text{even integrand}} dx = \pi \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a} = \frac{4}{3}\pi a^3$$

OR $$V = 2\pi \int_0^a (a^2 - x^2) dx = 2\pi \left[a^2 x - \frac{x^3}{3} \right]_0^a = \frac{4}{3}\pi a^3 \quad \#$$

HW EX (4) Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

Ans: 3π