Lecture (24)

Applications of Definite Integral
(I) Area

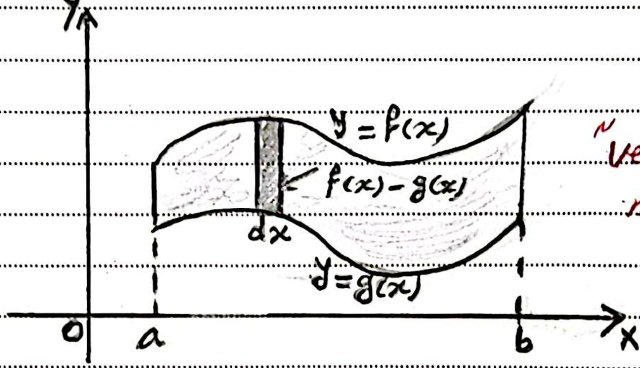
• Definition (1)

If f and g are continuous and $f(x) \geq g(x)$ $\forall x \in [a,b]$, then the area A of the region between the curves $y=f(x)$ and $y=g(x)$ from a to b is given by

$$A = \int_a^b [f(x) - g(x)]\,dx$$

Integration w.r.t x

limit of sum — length of a rectangle — width of a rectangle

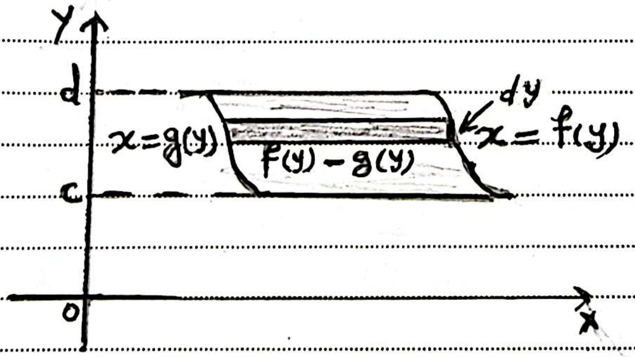


"vertical rectangles method"

• Similarly, the area A of the region between the curves $x=f(y)$ and $x=g(y)$ from c to d is given by

$$A = \int_c^d [f(y) - g(y)]\,dy$$

Integration w.r.t y



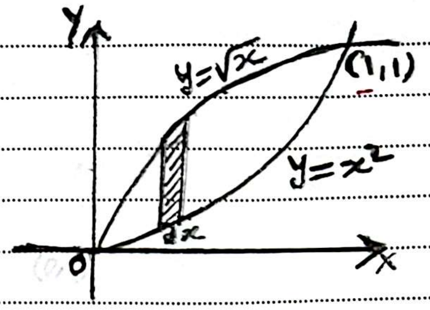
"Horizontal rectangles method"

• **Ex ①** Find the area of the region bounded by the graph of the equations: $y = x^2$ and $y = \sqrt{x}$.

Ans:

$y = x^2$ ①, $y = \sqrt{x}$ ②

To get the intersection points solve Eq. ① and Eq. ② simultaneously



$\Rightarrow x^2 = \sqrt{x}$

$\Rightarrow x^4 - x = 0$

$\Rightarrow x(x^3 - 1) = 0$

$\therefore x = 0, x = 1 \Rightarrow y = 0, y = 1$

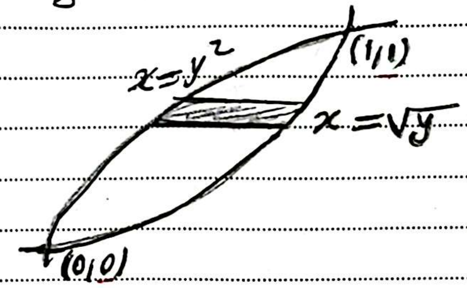
$\therefore$ The intersection points are $(0,0)$ and $(1,1)$

$A = \int_0^1 (\sqrt{x} - x^2)\,dx$ ... by using Vertical Res. Method.

$A = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ #

• Another Solution

$A = \int_0^1 (\sqrt{y} - y^2)\,dy$



$y = x^2$
$\Rightarrow x = \sqrt{y}$
& $y = \sqrt{x}$
$\Rightarrow x = y^2$

$A = \left[\frac{2}{3}y^{3/2} - \frac{1}{3}y^3\right]_0^1$

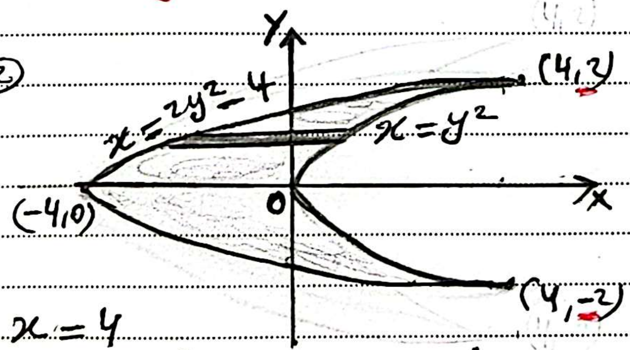
$\therefore A = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ #

• **Ex ②** Find the area of the region bounded by the graphs of the equations $2y^2 = x + 4$ and $y^2 = x$.

Ans:

$2y^2 = x + 4$ ①, $y^2 = x$ ②

The graphs of Eqs. ① and ② are two parabolas with the vertices $(-4, 0)$ and $(0,0)$.



the Vertex at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

Solving ① and ②,

$x + 4 = 2x \Rightarrow x = 4$

② $\Rightarrow y^2 = 4 \Rightarrow y = \pm 2$ $\therefore$ the intersection points are $(4,-2)$ and $(4,2)$.

$A = \int_{-2}^{2} [y^2 - (2y^2 - 4)]\,dy$ --- by using Horizontal Res. Method.

$\therefore A = \int_{-2}^{2} (-y^2 + 4)\,dy = 2\int_0^2 (4 - y^2)\,dy$ the integrand is even fn

$= 2\left[4y - \frac{1}{3}y^3\right]_0^2 = 2\left(8 - \frac{8}{3}\right) = \frac{32}{3}$ #

EX③ Find the area of the region bounded by the graphs of $y = x - 2$ and $y = -x^2 + 4x - 2$

Ans:

$y = x - 2$ ①, $y = -x^2 + 4x - 2$ ②

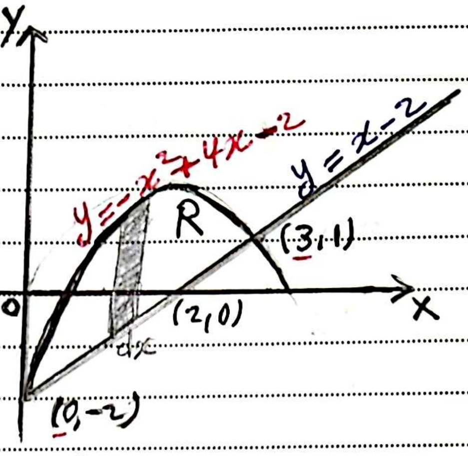
Solving ① and ②

$-x^2 + 4x - 2 = x - 2$

$\Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0$

$\therefore x = 0, x = 3 \Rightarrow y = -2, y = 1$

$\therefore$ the intersection points are $(0, -2)$ and $(3, 1)$.



$$A = \int_0^3 [(-x^2 + 4x - 2) - (x - 2)]\,dx$$

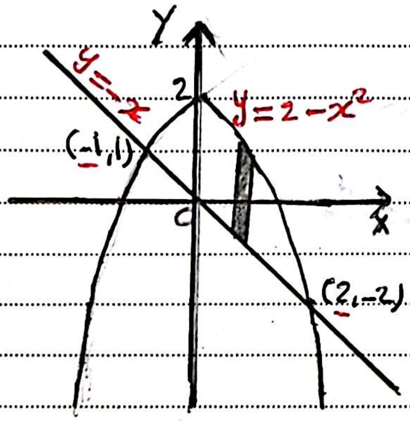
$$A = \int_0^3 (-x^2 + 3x)\,dx = \left[\frac{-x^3}{3} + \frac{3x^2}{2}\right]_0^3$$

$$\therefore A = \frac{-27}{3} + \frac{27}{2} = \frac{9}{2}$$ #

Ex ④ HW

Find the area bounded by the graphs of the equations $y = 2 - x^2$ and $y = -x$

Hint

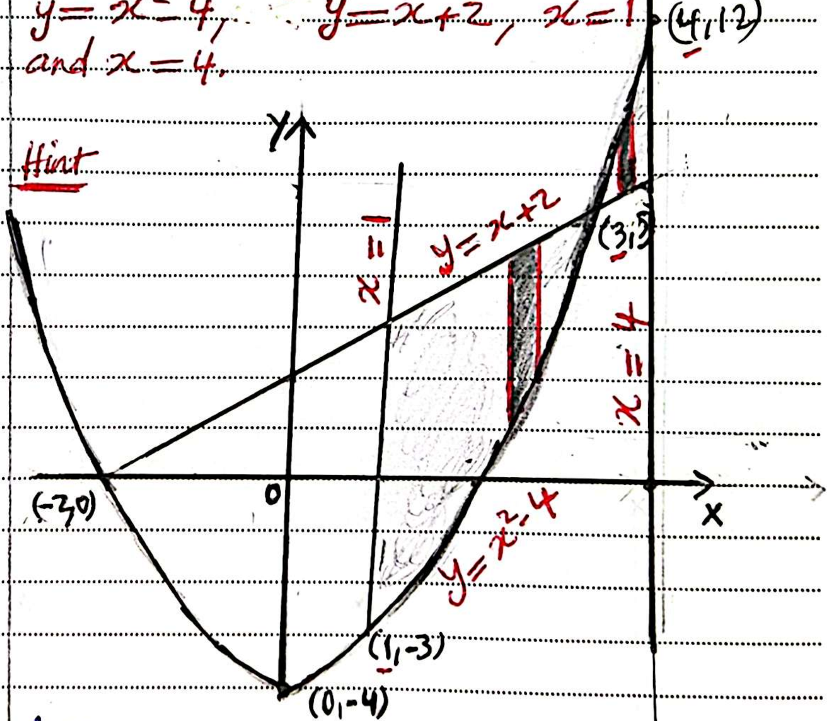


Ans: $\frac{9}{2}$

Ex⑤ HW

Find the area bounded by the graphs of the equations $y = x^2 - 4$, $y = x + 2$, $x = 1$ and $x = 4$.

Hint



Ans:

$$\text{Area} = \frac{22}{3} + \frac{17}{6} = \frac{61}{6}$$ #