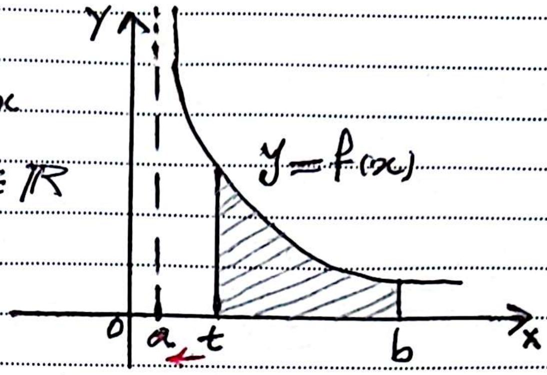
# Lecture (23) Improper Integrals (II)

**Definition**

Integrals of functions that become infinite at a point within the interval of Integration are called Improper Integrals of type II.

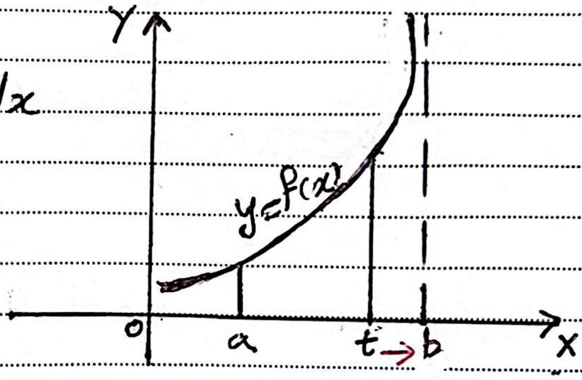
1. If $f$ is continuous on $(a,b)$ and discontinuous at $a$, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx \quad , t \in \mathbb{R}$$




2. If $f$ is continuous on $(a,b)$ and discontinuous at $b$, then

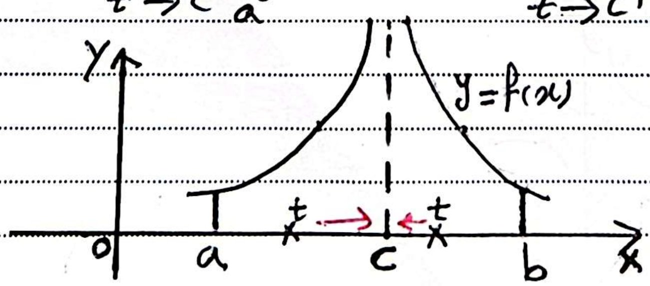
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$




3. If $f$ is discontinuous at $c$, where $a<c<b$, and continuous elsewhere on $[a,b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t\to c^-} \int_a^t f(x)\,dx + \lim_{t\to c^+} \int_t^b f(x)\,dx$$



• Determine wether the Integral converges or diverges and if it converges, find its value.

EX ① $\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$

Ans: $\int_0^{3} \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\int_0^t \frac{1}{\sqrt{3-x}}\,dx$

where the integrand is discontinuous at $x=3$.

$\therefore \int_0^3 \frac{1}{\sqrt{3-x}}\,dx = \lim_{t\to 3^-}\left[-\int_0^t \frac{-1}{\sqrt{3-x}}\,dx\right]$

$= \lim_{t\to 3^-}\left[-2\sqrt{3-x}\right]_0^t$

$\int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}+C$

$= \lim_{t\to 3^-}\left[-2\sqrt{3-t}+2\sqrt{3}\right]$

$= 2\sqrt{3}$

$\therefore$ The Imp. Integral is c'gt and its value is $2\sqrt{3}$.

H.W
EX ④ $\int_1^\infty \frac{\ln x}{x^2}\,dx$

Hint Use Integration by parts & L'Hopital Rule.

Ans: the Imp. Integral is c'gt, and its value is 1

EX⑤ $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx$

Ans: The Imp. Integral is c'gt and has the value $3+3\sqrt[3]{2}$.

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EX② $\int_0^1 \frac{1}{x}\,dx$

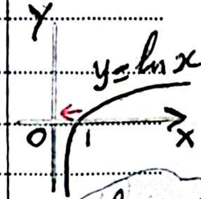
The integrand is discontinuous at $x=0$ so

$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx$

$\therefore \int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\left[\ln x\right]_t^1$

$= \lim_{t\to 0^+}(-\ln t) = \infty$

$\therefore$ The Imp. Integral is d'gt.




$\ln 1 = 0$
$\ln t \to -\infty$, as $t \to 0^+$

EX③ $\int_0^4 \frac{1}{(x-3)^2}\,dx$

The integrand is discontinuous at $x=3$, $3\in(0,4)$, Thus

$\int_0^4 \frac{1}{(x-3)^2}\,dx = \int_0^{3} \frac{1}{(x-3)^2}\,dx + \int_{3}^4 \frac{1}{(x-3)^2}\,dx$

$= \lim_{t\to 3^-}\int_0^t \frac{1}{(x-3)^2}\,dx + \lim_{t\to 3^+}\int_t^4 \frac{1}{(x-3)^2}\,dx$

$= \lim_{t\to 3^-}\left[\frac{-1}{x-3}\right]_0^t + \lim_{t\to 3^+}\left[\frac{-1}{x-3}\right]_t^4$

$= \lim_{t\to 3^-}\left[\frac{-1}{t-3}-\frac{1}{3}\right] + \lim_{t\to 3^+}\left[-1+\frac{1}{t-3}\right]$

$= \infty + \infty = \infty$, So the Imp. Integral is d'gt.

Note that: