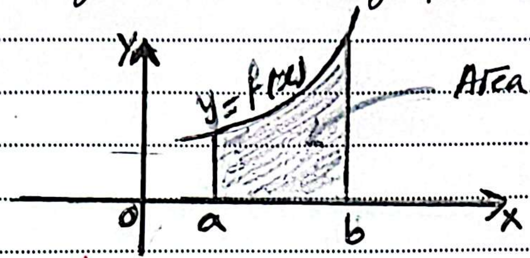
lecture (22) Improper Integrals

"التكاملات المعتلة"

If a function f is continuous and $f(x) \geq 0$ on $[a,b]$, then the definite Integral $\int_a^b f(x)\,dx$ is represented by the area of the region under the graph of f from a to b



$$\text{Area} = \int_a^b f(x)\,dx$$

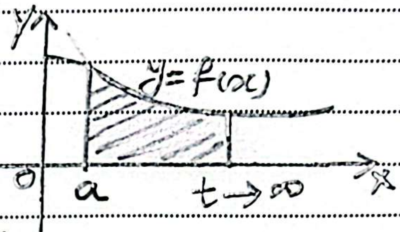
• Improper Integrals (Type I)

* Definition

Integrals with infinite limits of Integration are called Improper Integrals of type I.

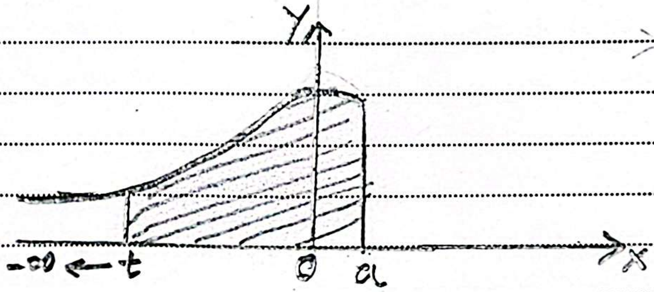
1. If f is continuous on $[a,\infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$



2. If f is continuous on $(-\infty, a]$, then

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty} \int_t^a f(x)\,dx$$



3. If f is continuous on $(-\infty,\infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$ where a is any real number

In each case if the limit exists and is finite we say that the Improper Integral Converges and that the limit is the value of the Improper Integral. If the limit doesn't exist or (equals) $\pm\infty$, the Improper Integral diverges.

- Determine wether the Integral converges or diverges and if it converges, find its value.

EX ① $\int_2^{\infty} \frac{1}{(x-1)^2}\,dx$ c'gt

Ans: $\int_2^{\infty} \frac{1}{(x-1)^2}\,dx$

$= \lim_{t\to\infty} \int_2^{t} \frac{1}{(x-1)^2}\,dx$

$= \lim_{t\to\infty} \left[\frac{-1}{x-1}\right]_2^t$

$= \lim_{t\to\infty} \left[\frac{-1}{t-1}+1\right] = 0+1 = 1$

Thus the given Improper Integral is convergent and its value is 1.

EX ② $\int_2^{\infty} \frac{1}{x-1}\,dx$ d'gt

Ans:

$\int_2^{\infty} \frac{1}{x-1}\,dx$

$= \lim_{t\to\infty} \int_2^{t} \frac{1}{x-1}\,dx$

$= \lim_{t\to\infty} \left[\ln(x-1)\right]_2^t$

$= \lim_{t\to\infty} [\ln(t-1) - \underbrace{\ln 1}_{0}] = \infty$

∴ The Improper Integral is divergent.

EX ③ $\int_{-\infty}^{1} e^{x}\,dx$ c'gt

Ans: $\int_{-\infty}^{1} e^{x}\,dx = \lim_{t\to-\infty} \int_t^{1} e^{x}\,dx$

$= \lim_{t\to-\infty} \left[e^{x}\right]_t^1$

$= \lim_{t\to-\infty} (e - e^{t})$

$\therefore \int_{-\infty}^{1} e^{x}\,dx = e - 0 = e$

∴ The Improper Integral is c'gt and has the value e.

EX ④ $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ c'gt

Ans:

$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_0^{\infty} \frac{1}{1+x^2}\,dx$

$= \lim_{t\to-\infty} \int_t^{0} \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_0^{t} \frac{1}{1+x^2}\,dx$

$= \lim_{t\to-\infty} \left[\tan^{-1}x\right]_t^0 + \lim_{t\to\infty} \left[\tan^{-1}x\right]_0^t$

$= \lim_{t\to-\infty} (0 - \tan^{-1}t) + \lim_{t\to\infty} (\tan^{-1}t - 0)$

$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$, the Imp. Integral is c'gt and has the value π.