Lecture (21) Indeterminate forms & L'Hôpital Rule

- Indeterminate forms such as

(I) $\frac{0}{0}$ and $\frac{\infty}{\infty}$ quotient forms

(II) $0\cdot\infty$, $0\cdot(-\infty)$, $\infty\cdot 0$ and $-\infty\cdot 0$ product forms

(III) $(-\infty)+\infty$ and $\infty-\infty$ Sum & difference forms

(IV) 0^0, 1^∞, $1^{-\infty}$ and ∞^0 exponential forms

- **L'Hôpital Rule**

Let f and g are differentiable functions on (a,b), $c\in(a,b)$ and if $\frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x=c$

then $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$, $g'(x)\neq 0$.

- For each of the following: Find the limit, if it exists.

(1) $\lim\limits_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25} \to \frac{0}{0}$

$\lim\limits_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25} = \lim\limits_{x\to 5}\frac{\frac{1}{2\sqrt{x-1}}}{2x}$ L'Hôpital Rule

$= \lim\limits_{x\to 5}\frac{1}{4x\sqrt{x-1}}$

$= \frac{1}{40}$

(2) $\lim\limits_{x\to \pi/2}\frac{1-\sin x}{\cos x} \to \frac{0}{0}$

$\lim\limits_{x\to \pi/2}\frac{1-\sin x}{\cos x}$ Apply L'Hôpital Rule

$= \lim\limits_{x\to \pi/2}\frac{-\cos x}{-\sin x} = \frac{\cos \pi/2}{\sin \pi/2}$

$= \frac{0}{1} = 0$

(3) $\lim\limits_{x\to \infty}\frac{x^2}{\ln x} \to \frac{\infty}{\infty}$

$\lim\limits_{x\to\infty}\frac{x^2}{\ln x} = \lim\limits_{x\to\infty}\frac{2x}{1/x}$ L'Hôpital Rule

$= \lim\limits_{x\to\infty} 2x^2 = \infty$

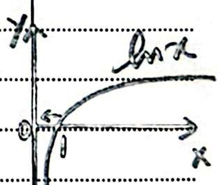
(4) $\lim\limits_{x\to 0^+}\sin x\,\ln(\sin x) \to 0\cdot(-\infty)$

$\lim\limits_{x\to 0^+}\sin x\,\ln(\sin x)$

$= \lim\limits_{x\to 0^+}\frac{\ln \sin x}{1/\sin x} \to \frac{-\infty}{\infty}$

$= \lim\limits_{x\to 0^+}\frac{(1/\sin x)\cos x}{(-1/\sin^2 x)\cos x}$ L'Hôpital Rule

$= \lim\limits_{x\to 0^+}(-\sin x) = 0$



$\lim\limits_{x\to\infty}\ln x = \infty$

$\lim\limits_{x\to 0^+}\ln x = -\infty$

(5) $\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right] \to \infty-\infty$

$\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]$

$= \lim_{x\to 0}\frac{x-\sin x}{x\sin x} \to \frac{0}{0}$

$= \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x}$ L'Hôpital Rule

$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x}$ " " "

$= \frac{0}{2} = 0$

* Note that:

- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$,
- $\lim_{h\to 0}(1+h)^{1/h} = e$
- $e \approx 2.71828\ldots$ is an irrational number.
- $\ln e = 1$

(6) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x} \to 1^\infty$

let $y=\left(1+\frac{1}{x}\right)^{5x}$

$\ln y = 5x\ln\left(1+\frac{1}{x}\right)$

$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}5x\ln\left(1+\frac{1}{x}\right) \to \infty\cdot 0$

$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{5\ln\left(1+\frac{1}{x}\right)}{1/x} \to \frac{0}{0}$

$\lim_{x\to\infty}\ln y = 5\lim_{x\to\infty}\frac{1/(1+1/x)\cdot\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{(-1/x^2)}}$

$\lim_{x\to\infty}\ln y = 5\lim_{x\to\infty}\frac{1}{1+1/x} = 5$

$\therefore \lim_{x\to\infty} y = e^5$

$\therefore \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5x} = e^5$ #

HW (7) $\lim_{x\to 1}\frac{x^3-3x+2}{x^2-2x+1} \to \frac{0}{0}$

Hint: Apply L'Hôpital's Rule twice.

Ans: 3

(8) $\lim_{x\to\infty} x\sin\frac{1}{x} \to \infty\cdot 0$ HW

Hint: $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{x\to\infty}\frac{\sin 1/x}{1/x} \to \frac{0}{0}$

Ans: 1

HW (9) $\lim_{x\to\infty} x^{1/x} \to \infty^0$

Hint: let $y = x^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln x$

Ans: 1

(10) $\lim_{x\to\infty}(x^2-1)e^{-x^2} \to \infty\cdot 0$ HW

Hint: $\lim_{x\to\infty}\frac{x^2-1}{e^{x^2}} \to \frac{\infty}{\infty}$

Ans: 0 #

* Note that:

$\lim_{x\to\infty} e^{-x} = 0$, $\lim_{x\to\infty} e^{x} = \infty$,

$\lim_{x\to-\infty} e^{-x} = \infty$, $\lim_{x\to-\infty} e^{x} = 0$