Lecture (20) Integrals involving quadratic or trig. expressions & Miscellaneous substitutions

• Evaluate the following Integrals.

① $\displaystyle\int \frac{1}{\sqrt{x^2+8x+25}}\,dx$

By using Complete square or perfect sq. we can write $x^2+8x+25$ as

$$x^2+8x+25 = x^2+8x+16+25-16 = (x+4)^2+9$$

$$\therefore I = \int \frac{1}{\sqrt{(x+4)^2+9}}\,dx$$

$$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \sinh^{-1}\left(\frac{u}{a}\right)+C$$

$$\therefore I = \sinh^{-1}\left(\frac{x+4}{3}\right)+C \quad (*)$$

OR you can use the substitution $x+4 = 3\tan\theta$ ...., the result is

$$I = \ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + C$$

which is equivalent to $(*)$

i.e. $I = \ln\left|\sqrt{x^2+8x+25}+x+4\right| + K$

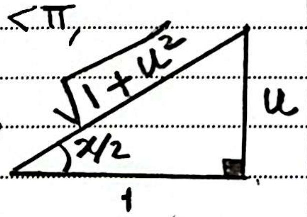
* If an integrand is a rational expression in $\sin x$ and $\cos x$ then use the following substitution

$u = \tan\left(\frac{x}{2}\right)$, $-\pi < x < \pi$,

$\Rightarrow dx = \frac{2}{1+u^2}\,du$

where $\sin x = \frac{2u}{1+u^2}$,

and $\cos x = \frac{1-u^2}{1+u^2}$



• Note that $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$,

$\cos x = \sqrt{1-\sin^2 x}$

② $\displaystyle\int \frac{1}{\sqrt{7+6x-x^2}}\,dx$

$$7+6x-x^2 = -(x^2-6x-7) = -(x-3)^2+9+7 = 16-(x-3)^2$$

$$\therefore I = \int \frac{1}{\sqrt{16-(x-3)^2}}\,dx$$

$$\therefore I = \sin^{-1}\left(\frac{x-3}{4}\right)+C$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right)+C$$

③ $\displaystyle\int \frac{1}{1+\sin x+\cos x}\,dx$

let $u = \tan(x/2)$, $dx = \frac{2}{1+u^2}\,du$

$\sin x = \frac{2u}{1+u^2}$,

$\cos x = \frac{1-u^2}{1+u^2}$

$$I = \int \frac{1}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du$$

$$= 2\int \frac{1}{1+\cancel{u^2}+2u+1-\cancel{u^2}}\,du$$

$$= \int \frac{1}{u+1}\,du = \ln|u+1|+C$$

$$= \ln\left|1+\tan(x/2)\right|+C.$$

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- If the integrand contains an expression of the form $\sqrt[n]{f(x)}$

③ $$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$$

we have $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$ in the integrand

$\because$ The least common denominator of $\frac{1}{2}$ and $\frac{1}{3}$ is 6

$\therefore$ We can use $u = x^{1/6}$

i.e. $x = u^6 \Rightarrow dx = 6u^5\,du$,
$\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$

$$\therefore I = \int \frac{6u^5}{u^3+u^2}\,du$$

$$I = 6\int \frac{u^5}{u^2(u+1)}\,du$$

$$I = 6\int \frac{u^3}{u+1}\,du$$

By long division,

$$\begin{array}{r|l} & u^2 - u + 1 \\ \hline u+1 & u^3 \\ & -u^3 \mp u^2 \\ \hline & -u^2 \\ & \pm u^2 \pm u \\ \hline & u \\ & -u \mp 1 \\ \hline & -1 \end{array}$$

$$I = 6\int \left[u^2 - u + 1 - \frac{1}{u+1}\right]du$$

$$I = 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + C$$

$$\therefore I = 2\sqrt{x} - 3\sqrt[3]{x} + 6x^{1/6} - 6\ln(x^{1/6}+1) + C$$

④ $$\int \frac{1}{\sqrt{e^x+1}}\,dx$$

let $u^2 = e^x + 1 \Rightarrow 2u\,du = e^x\,dx$

$\Rightarrow 2u\,du = (u^2-1)\,dx$

$\Rightarrow dx = \frac{2u}{u^2-1}\,du$

$$I = 2\int \frac{\cancel{u}}{\cancel{u}(u^2-1)}\,du$$

$$I = 2\int \frac{1}{(u-1)(u+1)}\,du$$

By using partial fractions,

$$\frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$

$\Rightarrow A(u+1) + B(u-1) = 1$

at $u = 1 \Rightarrow A = \frac{1}{2}$

at $u = -1 \Rightarrow B = -\frac{1}{2}$

$$\therefore I = \frac{\cancel{2}}{\cancel{2}}\int \left[\frac{1}{u-1} - \frac{1}{u+1}\right]du$$

$$I = \ln|u-1| - \ln|u+1| + C$$

$$I = \ln\left|\frac{u-1}{u+1}\right| + C$$

$$\therefore I = \ln\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right) + C$$

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⑤ Evaluate $\int \frac{2x-1}{x^2-6x+13}\,dx$ HW

Hint $x^2-6x+13 = (x-3)^2+4$, let $u = x-3$ ......

Ans: $I = \ln(x^2-6x+13) + \frac{5}{2}\tan^{-1}\left(\frac{x-3}{2}\right) + C$

⑥ Evaluate $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$ HW

Hint use $u = x^{1/6}$ ...... as ③

Ans: $I = \frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} + 2x^{1/2} - 6x^{1/6} + 6\tan^{-1}(x^{1/6}) + C.$