Chapter 2: Probability

Random Experiment:

Random experiment is an experiment we do not know its exact outcome in advance but we know the set of all possible outcomes.

Examples:

The following situations are random experiments:

- Tossing a coin has two possibilities, head (H) or Tail (T).
- Tossing a die and observe the number appears on top.
- A football team plays two games and in each game either he wins (W) or be equal (D) or losses (L).

The Sample Space

- The set of all possible outcomes of a statistical experiment is called the sample space and is denoted by S or $\boldsymbol{\Omega}$.

- Each outcome (element or member) of the sample space S is called a sample point.

If we are interested only in whether the number is even or odd, the sample space is simply

$$S=\{even,odd\}$$

Events

An event A is a subset of the sample space S. That is $A \subseteq S$.

- $\varphi \subseteq S$ is an event (φ is called the impossible event)
- $S \subseteq S$ is an event (S is called the sure event)

Example

Experiment: Selecting a ball from a box containing 6 balls numbered 1,2,3,4,5 and 6.

This experiment has 6 possible outcomes

$$S=\{1,2,3,4,5,6\}.$$

Consider the following events:

E_1=getting an event number $= \{2, 4, 6\} \subseteq \Omega$

E_2=getting a number less than 4 $= \{1, 2, 3\} \subseteq \Omega$

E_3=getting 1 or 3 $= \{1, 3\} \subseteq \Omega$

E_4=getting an odd number $= \{1, 3, 5\} \subseteq \Omega$

E_5=getting a negative number $= \{\ \} = \phi \subseteq \Omega$

E_6=getting a number less than 10

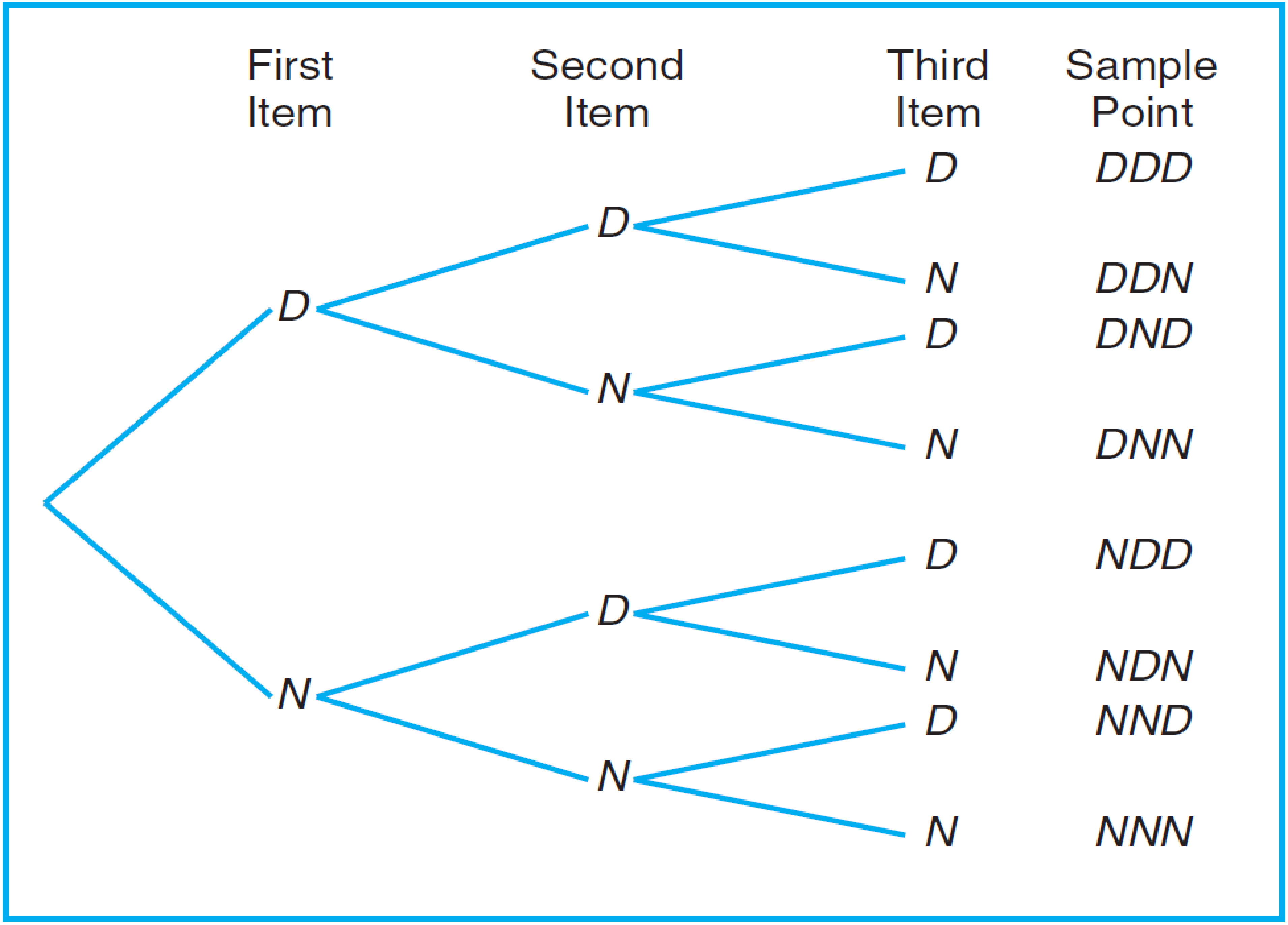
$$= \{1, 2, 3, 4, 5, 6\} = \Omega \subseteq \Omega$$

Notation

$$n(\Omega) = \text{no. of outcomes (elements) in } \Omega$$
$$n(E) = \text{no. of outcomes (elements) in the event } E$$

Example:

Experiment: Selecting 3 items from manufacturing process; each item is inspected and classified as defective (D) or non-defective (N).


First Item
Second Item
Third Item
Sample Point
D
D
D
DDD
N
DDN
N
D
DND
N
DNN
N
D
D
NDD
N
NDN
N
D
NND
N
NNN

This experiment has 8 possible outcomes

$$S = \{DDD, DDN, DND, DNN, NDD, NDN, NND, NNN\}$$

Consider the following events:

A={at least 2 defectives}=

{DDD,DDN,DND,NDD}$\subseteq S$

B={at most one defective}=

{DNN,NDN,NND,NNN}$\subseteq S$

C={3 defectives}= {DDD}$\subseteq S$

Rule (1)

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_{1\times}n_2$ ways.

Example

How many sample points are there in the sample space when a pair of dice is thrown once?

Solution:

The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways.

Therefore, the pair of dice can land in

$n_1 \times n_2 = (6)(6) = 36$ possible ways.

Rule (2)

If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 . n_2 \ldots n_k$ ways.

Example:

Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Solution:

Since $n_1 = 2$, $n_2 = 4$, $n_3 = 3$, and $n_4 = 5$, there are

$$n_l \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$$

different ways to order the parts.

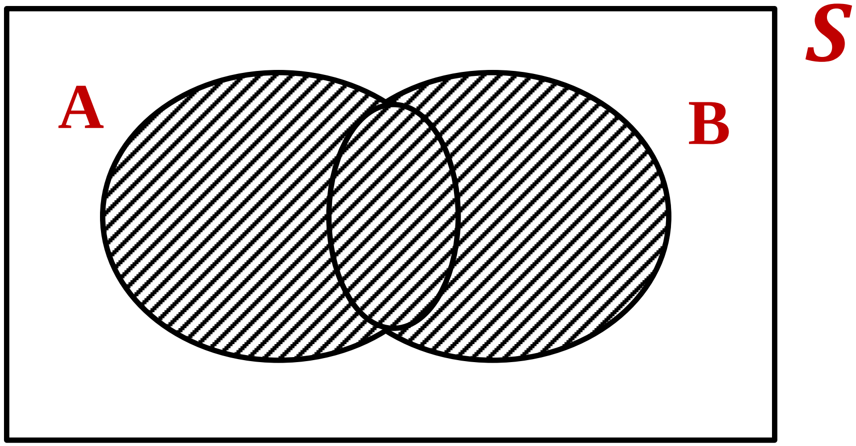
Some Operations on Events

Let A and B be two events defined on the sample space . S

Union: $A \cup B$

$\boldsymbol{A \cup B}$ Consists of all outcomes in A **or** in B **or** in both A and B.

$A \cup B = \{x \in S: x \in A \text{ or } x \in B\}$



- $A \cup B$ Occurs if A occurs, **or** B occurs, **or** both A and B occur.
- That is $A \cup B$ Occurs if at least one of A and B occurs.

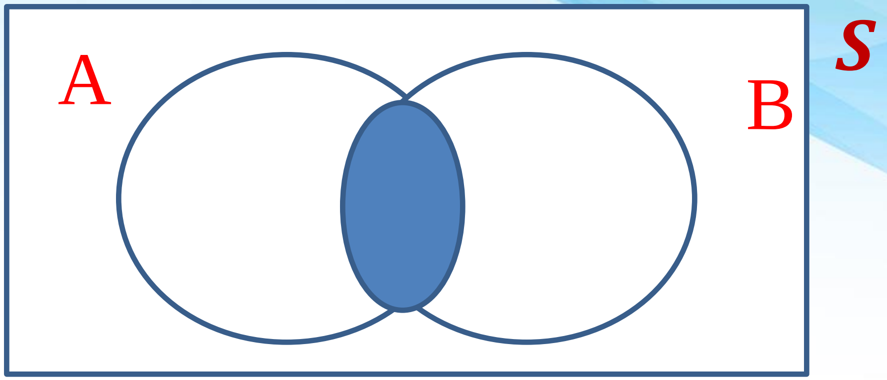
Example:

Let $A = \{a, b, c\}$ and $B = \{b, c, d, e\}$;

then $A \cup B = \{a, b, c, d, e\}$.

Intersection $A \cap B$

$A \cap B$ Consists of all outcomes in both *A* **and** *B*.



$A \cap B = AB = \{x \in S\text{: } x \in A \text{ and } x \in B\}$

$A \cap B$ Occurs if both *A* **and** *B* occur together.

Example:

Let $V = \{a, e, i, o, u\}$ and $C = \{l, r, s, t\}$;

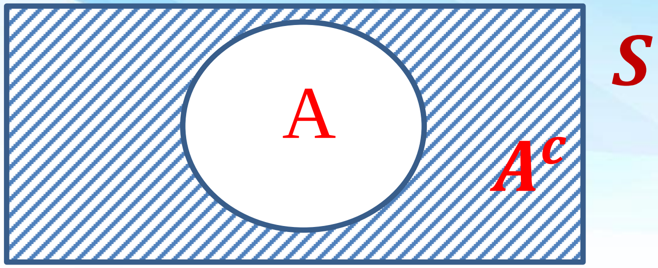
then $V \cap C = \varphi$.

Complement A^c

A^c is the complement of A.

A^c consists of all outcomes of S but are not in A.

A^c occurs if A does not.



$$\boldsymbol{A^c} \textbf{ or } \boldsymbol{A'} \textbf{ or } \overline{A}$$

$$A^c = \{x \in S: \ x \notin A \}$$

Example:

Let S = {1, 2, 3, 4, 5, 6}, *A*={1,2,4}

Then $A^c = \{3, 5, 6\}$

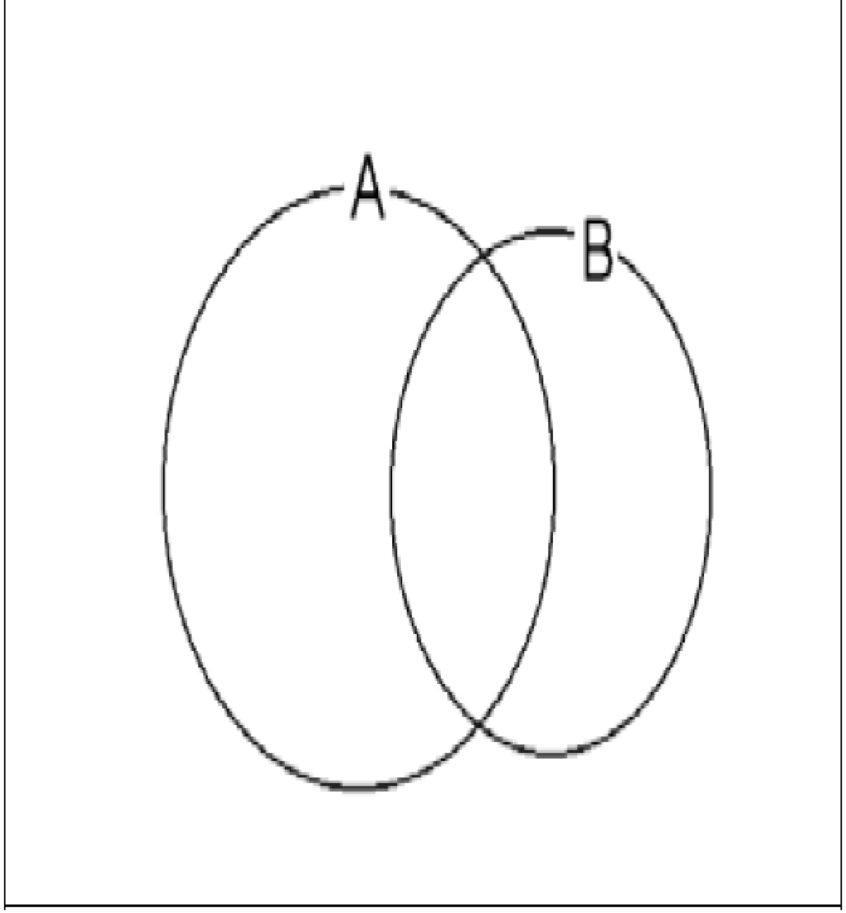
1. $A \cap \phi = \phi$.
2. $A \cup \phi = A$.
3. $A \cap A' = \phi$.
4. $A \cup A' = S$.
5. $S' = \phi$.
6. $\phi' = S$.
7. $(A')' = A$.
8. $(A \cap B)' = A' \cup B'$.
9. $(A \cup B)' = A' \cap B'$.

Exercises

2.3 – 2.4(a) – 2.7 and 2.14 on page 42

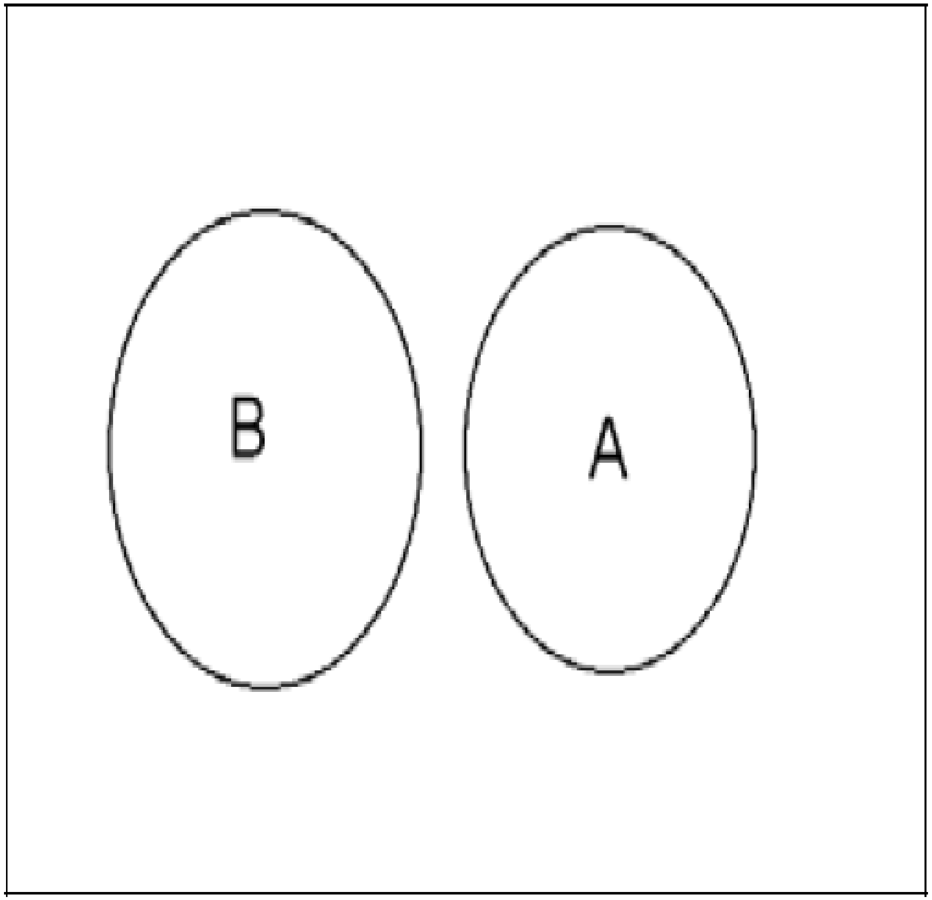
Mutually Exclusive (Disjoint) Events

Two events A and B are mutually exclusive (or disjoint) if and only if $A \cap B = \varphi$; that is, A and B have no common elements (they do not occur together).



$A \cap B \neq \phi$

A and *B* are not mutually exclusive



$A \cap B = \phi$

A and *B* are mutually exclusive (disjoint)

Combinations:

In many problems, we are interested in the number of ways of selecting r objects from n objects without regard to order. These selections are called combinations.

Combinations:

Notation: n factorial is denoted by $n!$ and is defined by:

$$n! = n(n-1)(n-2)\cdots(2)(1) \qquad for \quad n \geq 1$$

$$0! = 1$$

Example: $5! = (5)(4)(3)(2)(1) = 120$

Theorem:

The number of different ways for selecting r objects from n distinct objects is denoted by $\binom{n}{r}$ and is given by:

$$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}; \qquad r = 0, 1, 2, \ldots, n$$

Notes:

$\binom{n}{r}$ is read as " n " choose " r ".

$$\binom{n}{n} = 1 \qquad \binom{n}{0} = 1 \qquad \binom{n}{r} = \binom{n}{n-r}$$

Example

If we have 10 equal–priority operations and only 4 operating rooms, in how many ways can we choose the 4 patients to be operated on first?

Answer:

$n = 10 \qquad r = 4$

The number of different ways for selecting 4 patients from 10 patients is

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = \frac{10!}{4!\,6!} = \frac{(10)(9)(8)\cdots(2)(1)}{(4)(3)(2)(1)\quad(6)(5)(4)(3)(2)(1)}$$

$$= 210 \qquad\qquad (\textit{different ways})$$

Probability of an Event

- To every point (outcome) in the sample space of an experiment *S*, we assign a weight (or probability), ranging from 0 to 1, such that the sum of all weights (probabilities) equals 1.
- The weight (or probability) of an outcome measures its likelihood (chance) of occurrence.
- To find the probability of an event *A*, we sum all probabilities of the sample points in *A*. This sum is called the probability of the event *A* and is denoted by P(*A*).

Definition

The probability of an event A is the sum of the weights (probabilities) of all sample points in A. Therefore,

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. $P(\phi) = 0$

Example

A balanced coin is tossed twice. What is the probability that at least one head occurs?

Solution:

S = {HH, HT, TH, TT}

A = {at least one head occurs}= {HH, HT, TH}

Since the coin is balanced, the outcomes are equally likely; i.e., all outcomes have the same weight or probability.

Outcome	Weight (Probability)
HH	P(HH) = w
HT	P(HT) = w
TH	P(TH) = w
TT	P(TT) = w
sum	4w=1

$4w = 1 \Leftrightarrow w = 1/4 = 0.25$

$P(HH)=P(HT)=P(TH)=P(TT)=0.25$

$$P(A) = P(\{\text{at least one head occurs}\}) = P(\{HH, HT, TH\})$$
$$= P(HH) + P(HT) + P(TH)$$
$$= 0.25+0.25+0.25$$
$$= 0.75$$

Theorem

If an experiment has $n(S) = N$ equally likely different outcomes, then the probability of the event A is:

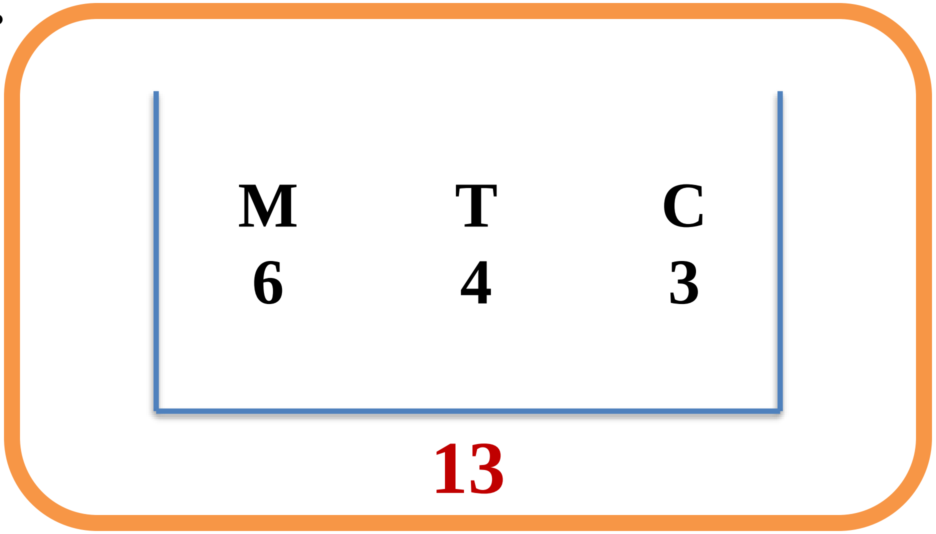
$$P(A) = \frac{n(A)}{n(S)} = \frac{n(A)}{N} = \frac{no.\ of\ outcomes\ \ in\ \ A}{no.\ of\ outcomes\ \ in\ \ S}$$

Example

A mixture of candies consists of 6 mints, 4 toffees, and 3 chocolates. If a person makes a random selection of one of these candies, find the probability of getting:

(a) a mint

(b) a toffee or chocolate.



Solution

Define the following events:

M = {getting a mint}

T = {getting a toffee}

C = {getting a chocolate}

Experiment: selecting a candy at random from 13 candies

$n(S) =$ no. of outcomes of the experiment of selecting a candy=13.

The outcomes of the experiment are equally likely because the selection is made at random.

(a) M = {getting a mint}

$n(M) = 6$

$$P(M) = P(\{\text{getting a mint}\}) = \frac{n(M)}{n(S)} = \frac{6}{13}$$

(b) $T \cup C$ = {getting a toffee or chocolate}

$n(T \cup C)$ = no. of different ways of selecting a toffee **or** a chocolate candy

= no. of different ways of selecting a toffee candy + no. of different ways of selecting a chocolate candy

= 4+3=7

$$P(T \cup C) = P(\{\text{getting a toffee or chocolate}\}) = \frac{n(T \cup C)}{n(S)} = \frac{7}{13}$$

Additive Rules:

Theorem:

If A and B are any two events, then:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Corollary 1:

If A and B are mutually exclusive (disjoint) events, then:

$$P(A \cup B) = P(A) + P(B)$$

Corollary 2:

If $A_1, A_2, \ldots, A_n$ are n mutually exclusive (disjoint) events, then:

$$\mathrm{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \ldots + \mathrm{P}(A_n)$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

Notes

1) $P(A) = P(A \cap B) + P(A \cap B^c)$

2) $P(B) = P(A \cap B) + P(A^c \cap B)$

3) $P(A \cap B^c) = P(A) - P(A \cap B)$

4) $P(A^c \cap B) = P(B) - P(A \cap B)$

5) $P(A^c \cap B^c) = 1 - P(A \cup B)$

6) $P(A \cup B) = P(A) + P(A^c \cap B)$

Example

The probability that Ahmad passes Mathematics is 2/3, and the probability that he passes English is 4/9. If the probability that he passes both courses is 1/4, what is the probability that he will:

(a) pass at least one course?

(b) pass Mathematics and fail English?

(c) fail both courses?

Solution:

Define the events:

M={Ahmed passes Mathematics}

E={Ahmed passes English}

We know that $P(M) = 2/3, P(E) = 4/9$, and $P(M \cap E) = 1/4.$

(a) Probability of passing at least one course is:

$$P(M \cup E) = P(M) + P(E) - P(M \cap E)$$

$$\frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}$$

(b) Probability of passing Mathematics and failing English is:

$$P(M \cap E^c) = P(M) - P(M \cap E)$$

$$\frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

(c) Probability of failing both courses is:

$$P(M^c \cap E^c) = 1 - P(M \cup E)$$

$$1 - \frac{31}{36} = \frac{5}{36}$$

Theorem

If A and A^c are complementary events, then:

$$P(A) + P(A^c) = 1$$

$$\Leftrightarrow P(A^c) = 1 - P(A)$$

Conditional Probability:

The probability of occurring an event A when it is known that some event B has occurred is called the conditional probability of A given B and is denoted $P(A|B)$.

Definition

The conditional probability of the event A given the event B is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad ; \; P(B) > 0$$

Notes:

1. $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} =$$

$$= \frac{n(A \cap B)/n(S)}{n(B)/n(S)} = \frac{n(A \cap B)}{n(B)}$$

2. $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

3. $$P(A \cap B) = P(A)\,P(B \mid A)$$

$$= P(B)\,P(A \mid B)$$

Example:

339 physicians are classified as given in the table below. A physician is to be selected at random.

(1) Find the probability that:

(a) the selected physician is aged 40 – 49

(b) the selected physician smokes occasionally

(c) the selected physician is aged 40 – 49 and smokes
occasionally

(2) Find the probability that the selected physician is aged
40 – 49 given that the physician smokes occasionally.

Smoking Habbit

Age

	Daily (B_1)	Occasionally (B_2)	Not at all (B_3)	Total
20-29 (A_1)	31	9	7	47
30-39 (A_2)	110	30	49	189
40-49 (A_3)	29	21	29	79
50+ (A_4)	6	0	18	24
Total	176	60	103	339

Solution:

$.n(S) = 339$ equally likely outcomes.

Define the following events:

A_3 = the selected physician is aged 40 – 49

B_2 = the selected physician smokes occasionally

$A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally

(1) (a) A_3 = the selected physician is aged 40 – 49

$$P(A_3) = \frac{n(A_3)}{n(S)} = \frac{79}{339} = 0.2330$$

(b) B_2 = the selected physician smokes occasionally

$$P(B_2) = \frac{n(B_2)}{n(S)} = \frac{60}{339} = 0.1770$$

(c) $A_3 \cap B_2$ = the selected physician is aged 40 – 49 and smokes occasionally.

$$P(A_3 \cap B_2) = \frac{n(A_3 \cap B_2)}{n(S)} = \frac{21}{339} = 0.06195$$

(2) $A_3|B_2$ = the selected physician is aged 40 – 49 given that the physician smokes occasionally

$$\text{(i)}\ P(A_3 \mid B_2) = \frac{P(A_3 \cap B_2)}{P(B_2)} = \frac{0.06195}{0.1770} = 0.35$$

$$\text{(ii)}\ P(A_3 \mid B_2) = \frac{n(A_3 \cap B_2)}{n(B_2)} = \frac{21}{60} = 0.35$$

(iii) We can use the restricted table directly:

$$P(A_3 \mid B_2) = \frac{21}{60} = 0.35$$

Note:

Notice that $P(A_3|B_2)=0.35 > P(A_3)=0.233$.

The conditional probability does not equal unconditional probability; i.e., $P(A_3|B_2) \neq P(A_3)$! What does this mean?

Note:

- $P(A|B) = P(A)$ means that knowing B has no effect on the probability of occurrence of A. In this case A is **independent** of B.
- $P(A|B) > P(A)$ means that knowing B increases the probability of occurrence of A.
- $P(A|B) < P(A)$ means that knowing B decreases the probability of occurrence of A.

Independent Events

Definition:

Two events A and B are **independent** if and only if $P(A|B) = P(A)$ and $P(B|A) = P(B)$. Otherwise A and B are **dependent**.

Example:

In the previous example, we found that

$$P(A_3/B_2) \neq P(A_3)$$

Therefore, the events A_3 and B_2 are dependent, i.e., they are not independent. Also, we can verify that

$$P(B_2/A_3) \neq P(B_2)$$

Multiplicative Rule

Theorem:

If $P(A) \neq 0$ and $P(B) \neq 0$, then:

$$P(A \cap B) = P(A)\,P(B|A)$$

$$= P(B)\,P(A|B)$$

Example

Suppose we have a fuse box containing 20 fuses of which 5 are defective (D) and 15 are non-defective (N). If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Solution:

Define the following events:

A = {the first fuse is defective}

B = {the second fuse is defective}

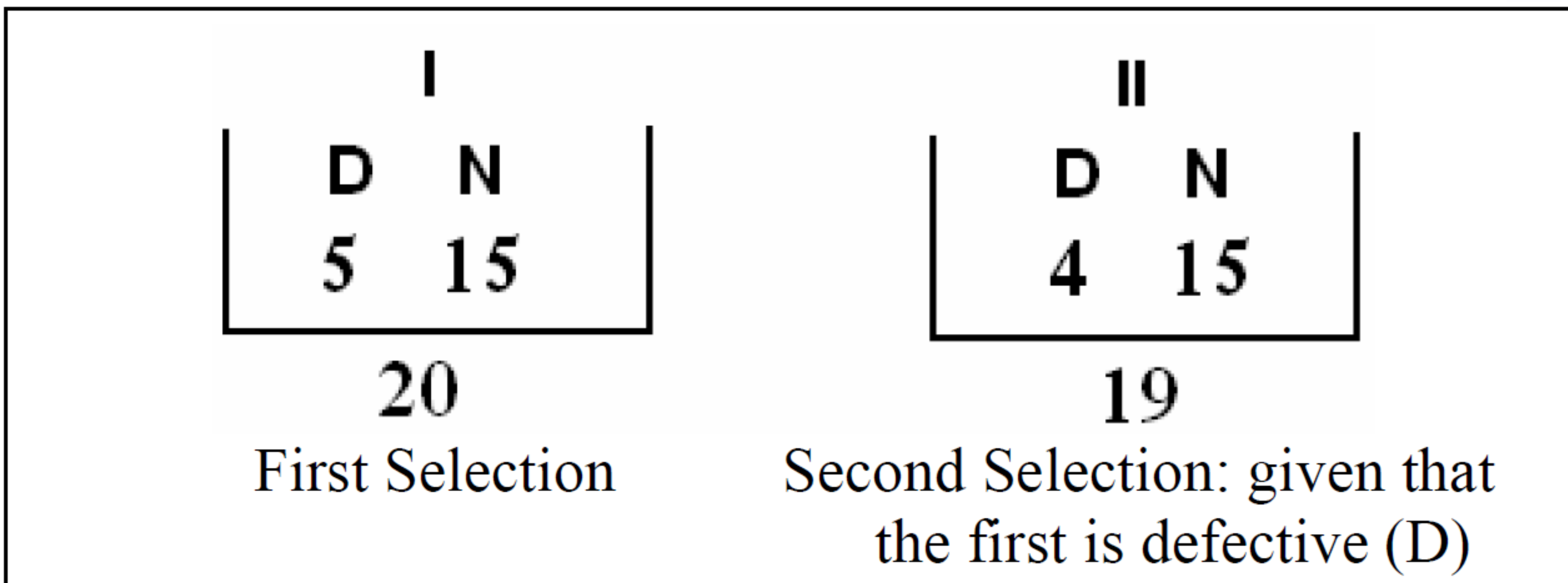
$A \cap B$ ={the first fuse is defective and the second fuse is defective} = {both fuses are defective}

We need to calculate $P(A \cap B)$.

$$P(A) = \frac{5}{20}$$

$$P(B|A) = \frac{4}{19}$$

$$P(A \cap B) = P(A)\, P(B|A)$$
$$= \frac{5}{20} \times \frac{4}{19} = 0.052632$$



Theorem

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)\,P(B)$$

(Multiplicative Rule for independent events)

Note:

Two events A and B are independent if one of the following conditions is satisfied:

(i) $P(A|B) = P(A)$

$\Leftrightarrow$ (ii) $P(B|A) = P(B)$

$\Leftrightarrow$ (iii) $P(A \cap B) = P(A)\,P(B)$

Theorem $(k = 3)$

If A_1, A_2, A_3 are 3 events, then:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2 | A_1)\,P(A_3 | A_1 \cap A_2)$$

If A_1, A_2, A_3 are 3 independent events, then:

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$$

Bayes' Rule

Definition:

The events A_1, A_2,…, and A_n constitute a partition of the sample space S if:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = S$$

$$A_i \cap A_j = \emptyset, \qquad \forall \; i \neq j$$

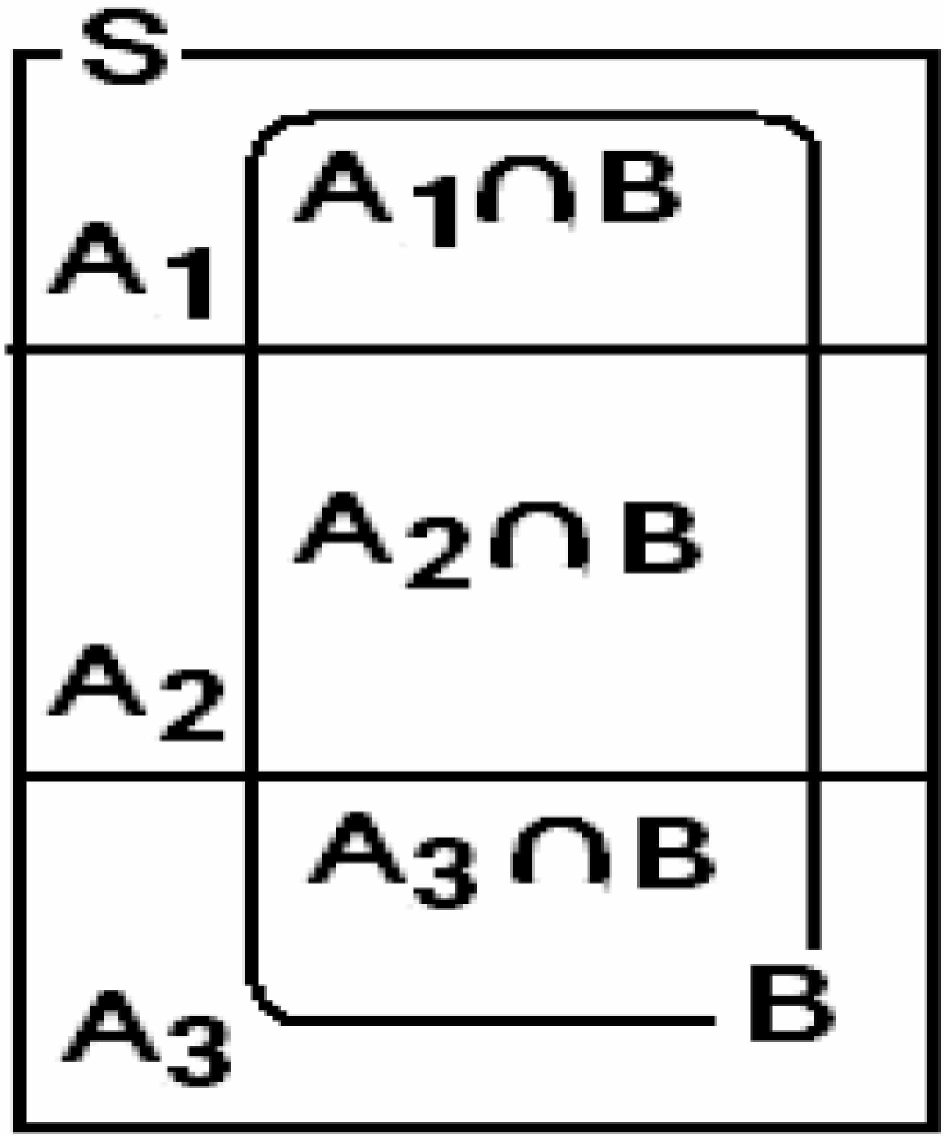
Theorem: (Total Probability)

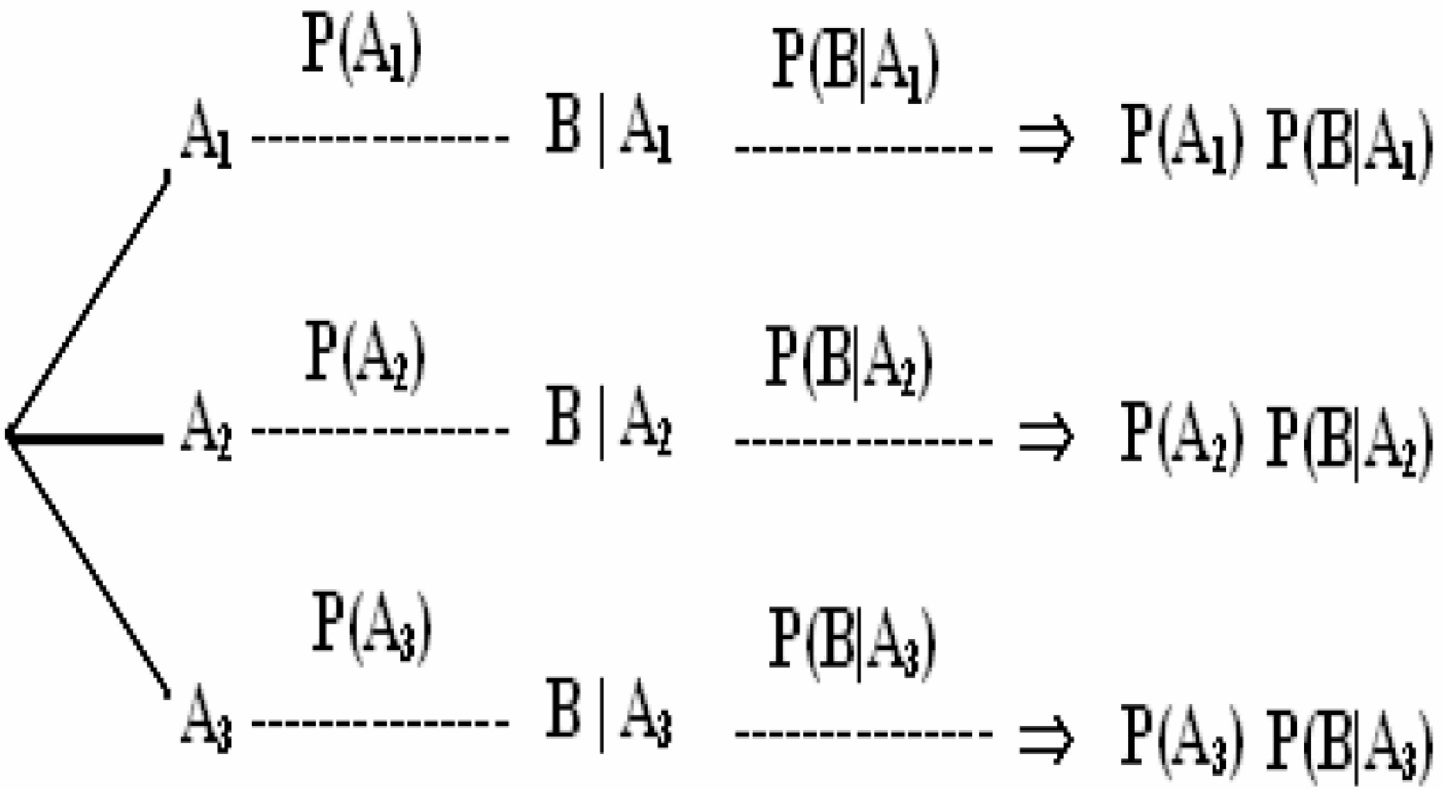
If the events A_1, A_2,…, and A_n constitute a partition of the sample space S such that

$$P(A_k) \neq 0 \text{ for } k = 1,2,\dots,n,$$

then for any event B:

$$P(B) = \sum_{k=1}^{n} P(A_k \cap B) = \sum_{k=1}^{n} P(A_k).P(B/A_k)$$


S
A1
A1∩B
A2∩B
A2
A3∩B
B
A3



$$\text{المجموع} = P(B) = \sum_{k=1}^{n} P(A_k)\, P(B \mid A_k)$$

Tree Diagram

Example

Three machines A_1, A_2 and A_3 make 20%, 30%, and 50%, respectively, of the products. It is known that 1%, 4%, and 7% of the products made by each machine, respectively, are defective. If a finished product is randomly selected, what is the probability that it is defective?

Solution:

Define the following events:

B = {the selected product is defective}

A_1 = {the selected product is made by machine A_1}

A_2= {the selected product is made by machine A_2}

A_3 = {the selected product is made by machine A_3}

$$P(A_1) = \frac{20}{100} = 0.2; \quad P(B|A_1) = \frac{1}{100} = 0.01$$

$$P(A_2) = \frac{30}{100} = 0.3; \quad P(B|A_2) = \frac{4}{100} = 0.04$$

$$P(A_3) = \frac{50}{100} = 0.5; \quad P(B|A_3) = \frac{7}{100} = 0.07$$

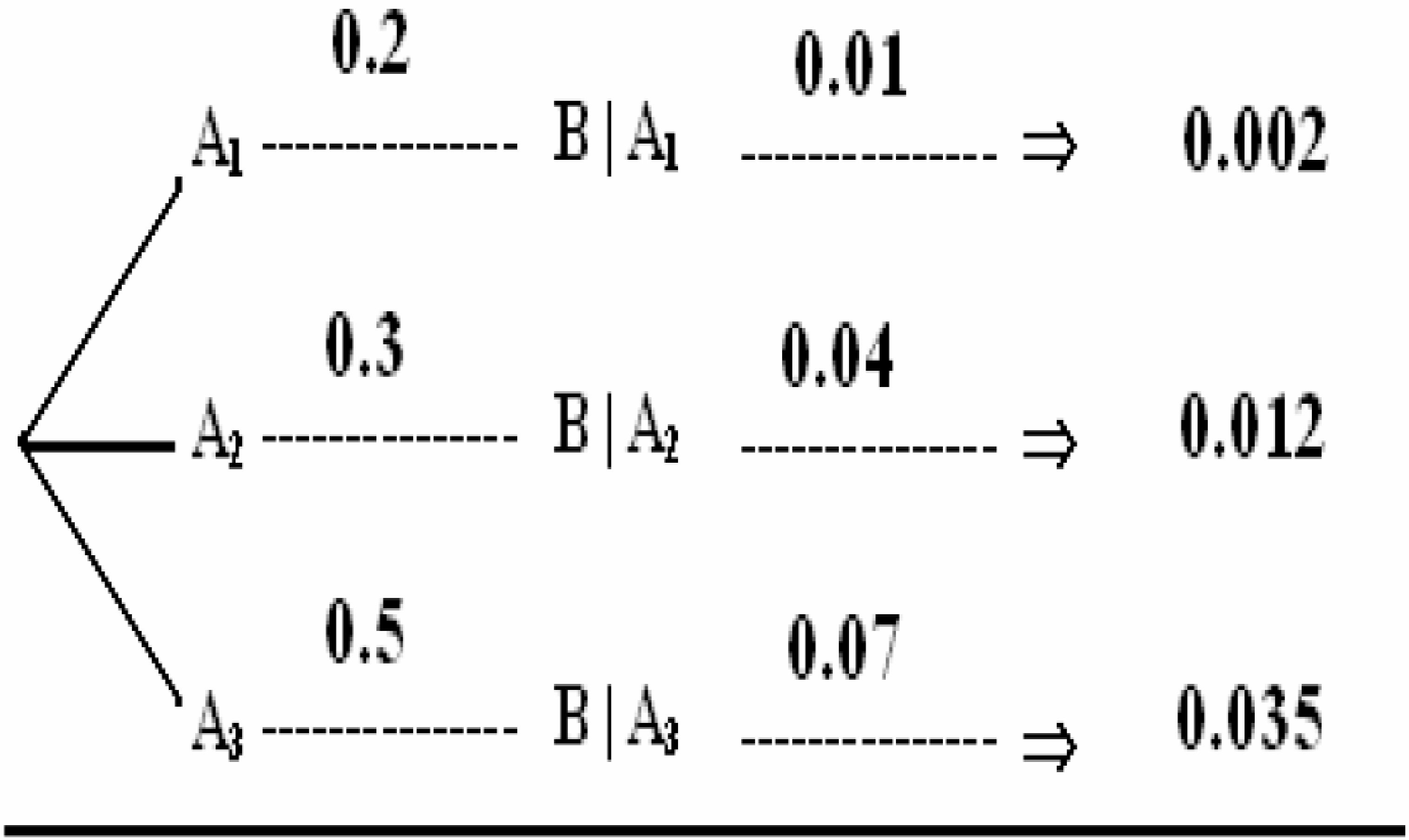
$$P(B) = \sum_{k=1}^{3} P(A_k)\,P(B \mid A_k)$$

$$= P(A_1)\,P(B|A_1) + P(A_2)\,P(B|A_2) + P(A_3)\,P(B|A_3)$$

$$= 0.2\times0.01 + 0.3\times0.04 + 0.5\times0.07$$

$$= 0.002 + 0.012 + 0.035$$

$$= 0.049$$



المجموع $= P(B) =$ 0.049

Another Example: see Example 2.41 page 74

Question:

If it is known that the selected product is defective, what is the probability that it is made by machine A_1?

Answer:

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{P(A_1)P(B|A_1)}{P(B)} = \frac{0.2 \times 0.01}{0.049} = \frac{0.002}{0.049} = 0.0408$$

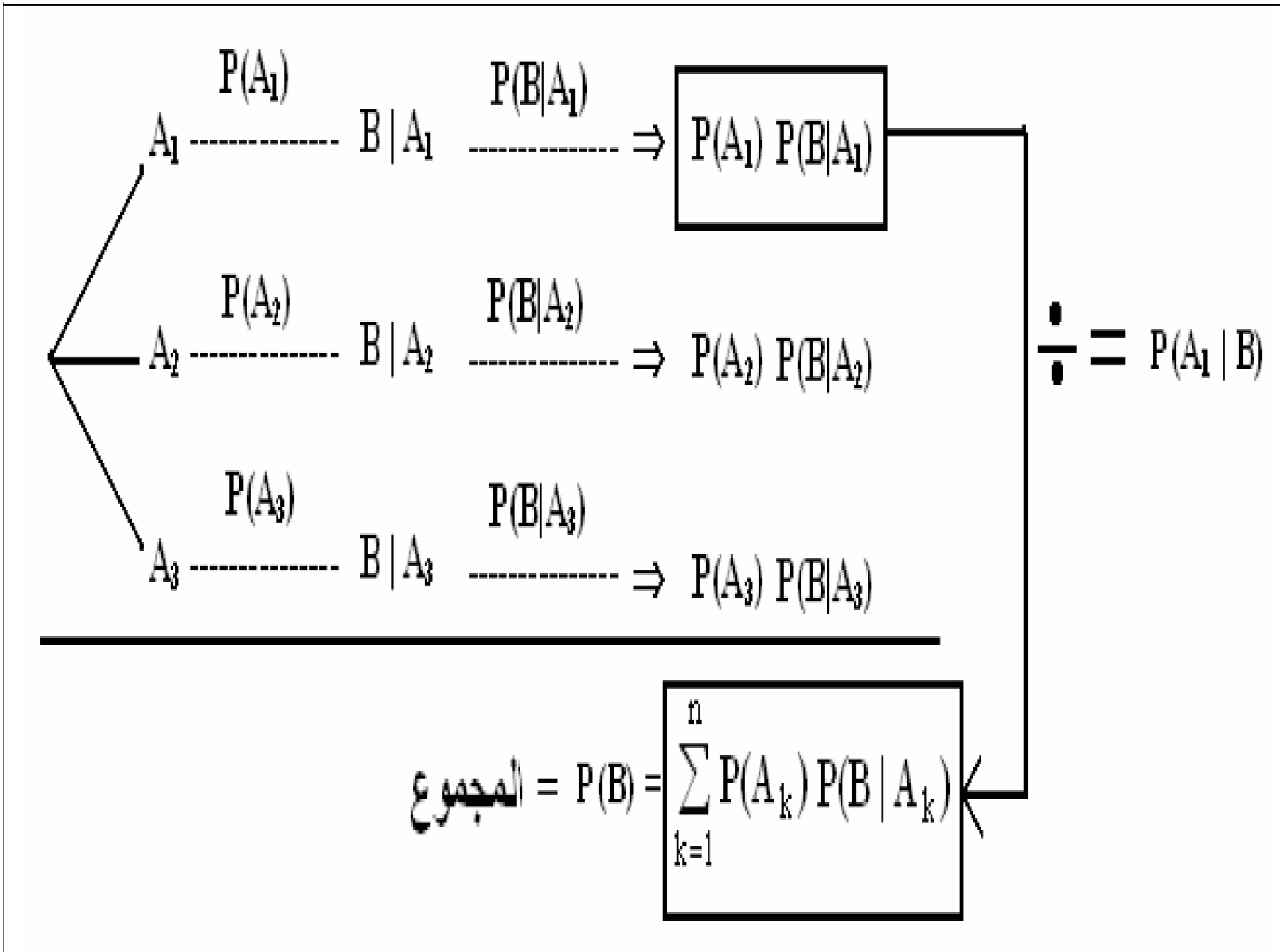
This rule is called Bayes' rule.

Theorem: (Bayes' rule)

If the events $A_1, A_2, \ldots,$ and A_n constitute a partition of the sample space S such that $P(A_k) \neq 0$ for $k = 1, 2, \ldots, n$, then for any event B such that $P(B) \neq 0$:

$$P(A_i \mid B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{k=1}^{n} P(A_k)P(B \mid A_k)} = \frac{P(A_i)P(B \mid A_i)}{P(B)}$$

for $i = 1, 2, \ldots, n$.



P(A1)
A1 B|A1
P(B|A1)
P(A1) P(B|A1)
P(A2)
A2 B|A2
P(B|A2)
P(A2) P(B|A2)
÷ = P(A1 | B)
P(A3)
A3 B|A3
P(B|A3)
P(A3) P(B|A3)
المجموع = P(B) = $\sum_{k=1}^{n} P(A_k) P(B | A_k)$

Example

In the previous example, if it is known that the selected product is defective, what is the probability that it is made by:

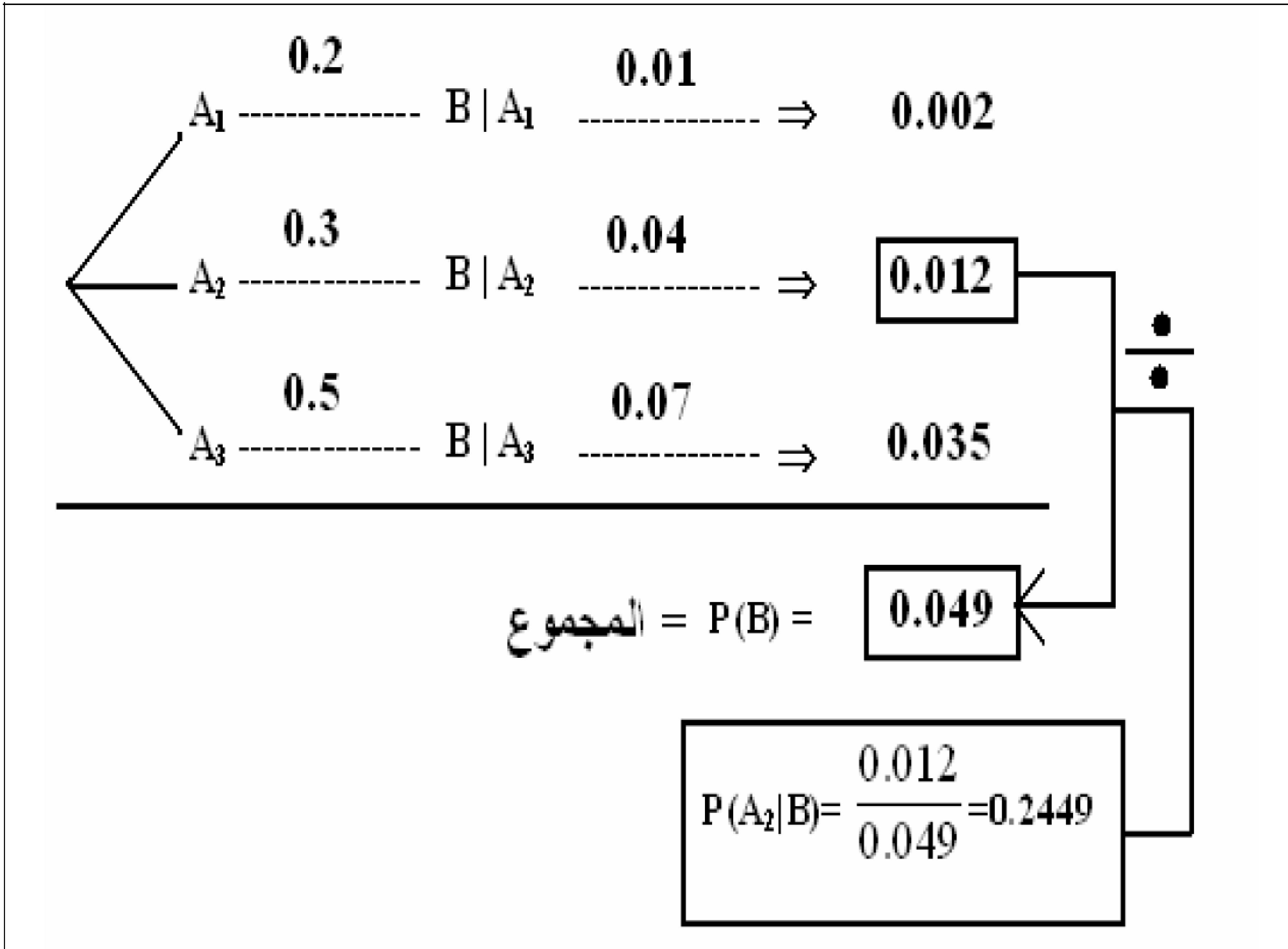
(a) machine A_2?

(b) machine A_3?

Solution:

$$\text{(a) } P(A_2|B) = \frac{P(A_2)P(B|A_2)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \frac{P(A_2)P(B|A_2)}{P(B)}$$

$$= \frac{0.3 \times 0.04}{0.049} = \frac{0.012}{0.049} = 0.2449$$


0.2
0.01
A1
B | A1
0.002
0.3
0.04
A2
B | A2
0.012
0.5
0.07
A3
B | A3
0.035
المجموع = P(B) =
0.049
P(A2|B)= 0.012 / 0.049 =0.2449

$$(b)\ P(A_3|B) = \frac{P(A_3)P(B|A_3)}{\sum_{k=1}^{n} P(A_k)P(B|A_k)} = \frac{P(A_3)P(B|A_3)}{P(B)}$$

$$= \frac{0.5 \times 0.07}{0.049} = \frac{0.035}{0.049} = 0.7142$$

Note:

P($A_1|B$) = 0.0408, P($A_2|B$) = 0.2449, P($A_3|B$) = 0.7142

$$\sum_{k=1}^{3} P(A_k/B) = 1$$

If the selected product was found defective, we should check machine A_3 first, if it is ok, we should check machine A_2, if it is ok, we should check machine A_1.

Another Example: see Example 2.42 page 75