Lecture (19) Integration of Rational Functions by Partial Fractions (Continued)

To evaluate $\int q(x)\,dx$ where $q(x) = \dfrac{f(x)}{g(x)}$

- We have the two following cases besides the usual one that studied before in L(18) where $g(x)$ is factorized as a product of different linear factors

Case ① If $g(x)$ of the form $(ax+b)^n$, $n \geq 1$ (repeated irreducible linear factors) then we can write $q(x)$ as follows.

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$$q(x) = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots$$

Case ② If $g(x)$ of the form $(ax^2+bx+c)^n$, $n \geq 1$ (repeated irreducible quadratic factors) then we can write $q(x)$ as follows.

$$q(x) = \frac{A_1 x + B_1}{ax^2+bx+c} + \frac{A_2 x + B_2}{(ax^2+bx+c)^2} + \cdots$$

EX① Evaluate $\displaystyle\int \frac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3}\,dx$

Ans: $q(x) = \dfrac{3x^3 - 18x^2 + 29x - 4}{(x+1)(x-2)^3} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2} + \dfrac{D}{(x-2)^3}$

$\Rightarrow A(x-2)^3 + B(x+1)(x-2)^2 + C(x+1)(x-2) + D(x+1)$

$= 3x^3 - 18x^2 + 29x - 4 \quad (*)$

at $x = 2 \Rightarrow 3D = 6 \quad \therefore D = 2$

at $x = -1 \Rightarrow -27A = -54 \quad \therefore A = 2$

To get B, equating the coefficients of x^3 for both sides of Eq $(*)$

$A + B = 3 \Longrightarrow 2 + B = 3 \quad \therefore B = 1$

To get C, equating the constant terms of both sides of Eq. $(*)$ (i.e. $x = 0$)

$-8A + 4B - 2C + D = -4 \Rightarrow -16 + 4 - 2C + 2 = -4$

$\therefore C = -3$

$$\therefore q(x) = \frac{2}{x+1} + \frac{1}{x-2} - \frac{3}{(x-2)^2} + \frac{2}{(x-2)^3}$$

$$I = \int \left[\frac{2}{x+1} + \frac{1}{x-2} - \frac{3}{(x-2)^2} + \frac{2}{(x-2)^3} \right] dx$$

$$I = 2\ln|x+1| + \ln|x-2| - \frac{3(x-2)^{-1}}{-1} + \frac{2(x-2)^{-2}}{-2} + C$$

$$\therefore I = \ln\left|(x-2)(x+1)^2\right| + \frac{3}{x-2} - \frac{1}{(x-2)^2} + C$$

EX (2) Use partial fractions to evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

$$q(x) = \frac{-2x+4}{(x^2+1)(x-1)^2}$$

$$= \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$\Rightarrow (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) = -2x+4 \quad (*)$

at $x=1 \Rightarrow 2D = 2 \quad \therefore D = 1$ (1)

Equating the Coefficients of x^3 of Eq. $(*)$

$A + C = 0$ (2)

Equating the Coefficients of x^2 of Eq. $(*)$

$-2A + B - C + D = 0$ (3)

Equating the Coefficients of x^0 (constant terms) of Eq. $(*)$ $x=0$

i.e. at $x=0 \Rightarrow B - C + D = 4$ (4)

Solving (1), (2), (3) and (4) we get $A=2$, $C=-2$, $D=1$, $B=1$

i.e. $A=2$, $B=1$, $C=-2$, $D=1$

$$\therefore q(x) = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}$$

$$\therefore I = \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$$

$$\therefore I = \int \left[\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right] dx$$

$$I = \int \left[\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right] dx$$

$$I = \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C$$

(H.W) Use partial fractions to evaluate the following Integrals.

Q1 $\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx$ Revise L(18)

Ans: $I = x^2 + 2\ln|x+1| + 3\ln|x-3| + C$

$I = x^2 + \ln\left|(x+1)^2(x-3)^3\right| + C$

Q2 $\displaystyle\int \frac{6x+7}{(x+2)^2}\,dx$

Hint: $\displaystyle\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$

Ans: $\displaystyle I = 6\ln|x+2| + \frac{5}{x+2} + C$

Q3 $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$

Hint: $\displaystyle\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

Ans: $\displaystyle I = \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C$

i.e. $\displaystyle I = \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + C$

Q4 $\displaystyle\int \frac{x^2-x-21}{2x^3-x^2+8x-4}\,dx$

Hint $\displaystyle\frac{x^2-x-21}{2x^3-x^2+8x-4} = \frac{x^2-x-21}{(x^2+4)(2x-1)}$

$\displaystyle = \frac{Ax+B}{x^2+4} + \frac{C}{2x-1}$

Ans:

$\displaystyle I = \frac{3}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{5}{2}\ln|2x-1| + C$

Q5 $\displaystyle\int \frac{4x+1}{(x-3)(x^2+6x+12)}\,dx$

Hint $\displaystyle\frac{4x+1}{(x-3)(x^2+6x+12)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+6x+12}$

Ans:

$\displaystyle I = \frac{1}{3}\ln|x-3| - \frac{1}{6}\ln|x^2+6x+12| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C$

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