Lecture (18) IV Integration of Rational Functions by partial Fractions

Evaluating $\int q(x)\,dx$

where q is a rational function of the form $q(x)=\dfrac{f(x)}{g(x)}$,

$f(x)$ and $g(x)$ are two polynomials.

EX 1

- Evaluate $\displaystyle\int \frac{x^3+3x-2}{x^2-x}\,dx$ by using partial Fractions.

Step 1 the degree of numerator > the degree of denominator

$\Rightarrow$ use long Division

$$q(x)=\frac{x^3+3x-2}{x^2-x}=x+1+\frac{4x-2}{x^2-x} \quad (1)$$

$$\begin{array}{r|l} & x+1 \\ \hline x^2-x & x^3 \qquad +3x-2 \\ & -x^3 \pm x^2 \\ \hline & x^2+3x-2 \\ & -x^2 \pm x \\ \hline & 4x-2 \end{array}$$

Step 2

Factorize the denominator into irreducible factors

$\Rightarrow$

$$q(x)=x+1+\frac{4x-2}{x(x-1)} \quad (2)$$

Step 3 Write the rational fn by using partial fractions as

$$q(x)=x+1+\frac{A}{x}+\frac{B}{x-1} \quad (3)$$

(2), (3) $\Rightarrow \dfrac{A}{x}+\dfrac{B}{x-1}=\dfrac{4x-2}{x(x-1)}$

$\therefore \dfrac{A(x-1)+Bx}{x(x-1)}=\dfrac{4x-2}{x(x-1)}$

$\Rightarrow A(x-1)+Bx=4x-2$

at $x=1 \Rightarrow B=2$

at $x=0 \Rightarrow -A=-2 \Rightarrow A=2$

(3) $\Rightarrow$

$$\therefore q(x)=x+1+\frac{2}{x}+\frac{2}{x-1} \quad (4)$$

Step 4 Evaluate the Integral by using (4)

$$I=\int\frac{x^3+3x-2}{x^2-x}\,dx$$

$$I=\int\left[x+1+\frac{2}{x}+\frac{2}{x-1}\right]dx$$

$$I=\frac{1}{2}x^2+x+2\ln|x|+2\ln|x-1|+c$$

$$I=\frac{1}{2}x^2+x+\ln x^2(x-1)^2+c$$

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EX 2

Evaluate $\displaystyle\int\frac{5x-3}{x^2-2x-3}\,dx$

HW

Ans:

$$I=2\ln|x+1|+3\ln|x-3|+c$$

Ex ③ Evaluate $\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$

Ans: $q(x) = \frac{4x^2+13x-9}{x(x^2+2x-3)}$

$q(x) = \frac{4x^2+13x-9}{x(x+3)(x-1)}$ ①

For the factor $x \Rightarrow \frac{A}{x}$

" " " $x+3 \Rightarrow \frac{B}{x+3}$

" " " $x-1 \Rightarrow \frac{C}{x-1}$

$\Rightarrow q(x) = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1}$ ②

①, ② $\Rightarrow A(x+3)(x-1) + Bx(x-1) + Cx(x+3) = 4x^2+13x-9$ ③

We find the coefficients A, B and C as follows.

* let $x=0$ in ③ $\Rightarrow -3A = -9 \Rightarrow \therefore A = 3$,
* let $x=1$ in ③ $\Rightarrow 4C = 8 \Rightarrow \therefore C = 2$,
* let $x=-3$ in ③ $\Rightarrow 12B = -12 \Rightarrow \therefore B = -1$

Note that in the previous Examples the denominator of the rational function is a product of different linear factors (not repeated).

$\therefore q(x)$ in ② becomes, $q(x) = \frac{3}{x} - \frac{1}{x+3} + \frac{2}{x-1}$

$\therefore I = \int \left[\frac{3}{x} - \frac{1}{x+3} + \frac{2}{x-1}\right] dx$

$I = 3\ln|x| - \ln|x+3| + 2\ln|x-1| + c$

$\therefore I = \ln\left|\frac{x^3(x-1)^2}{x+3}\right| + c$

Ex ④

Evaluate $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)}\,dx$ HW

Ans: $I = \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + c$