Lecture 17 (III) Integrals by using Trigonometric Substitutions

• Expression in integrand		Trigonometric substitutions
① $\sqrt{a^2-x^2}$	$\Rightarrow$	$x = a\sin\theta$
② $\sqrt{a^2+x^2}$	$\Rightarrow$	$x = a\tan\theta$
③ $\sqrt{x^2-a^2}$	$\Rightarrow$	$x = a\sec\theta$

• Evaluate the following Integrals

① $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$

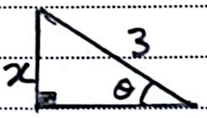
let $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$

$9 - 9\sin^2\theta = 9\cos^2\theta$

$\Rightarrow \sqrt{9-x^2} = 3\cos\theta$

$I = \displaystyle\int \frac{1}{\cancel{3\cos\theta}}\,\cancel{3\cos\theta}\,d\theta = \int d\theta$

$\therefore I = \theta + C$



$I = \sin^{-1}\left(\frac{x}{3}\right) + C$ $\qquad \sin\theta = \frac{x}{3}$

You can solve directly as

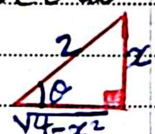
$\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + C$

③ $\displaystyle\int \frac{1}{x\sqrt{4-x^2}}\,dx$

let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$

$I = \displaystyle\int \frac{\cancel{2\cos\theta}}{2\sin\theta\, \cancel{2\cos\theta}}\,d\theta = \frac{1}{2}\int \csc\theta\, d\theta$

$I = \frac{1}{2}\ln|\csc\theta - \cot\theta| + C$



$I = \frac{1}{2}\ln\left[\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right] + C$

② $\displaystyle\int \frac{1}{(x^2-1)^{3/2}}\,dx$

let $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$

$I = \displaystyle\int \frac{1}{(\tan^2\theta)^{3/2}}\sec\theta\tan\theta\, d\theta$

$I = \displaystyle\int \frac{1}{\tan^3\theta}\sec\theta\tan\theta\, d\theta$

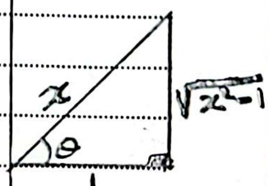
$I = \displaystyle\int \frac{1}{\tan^2\theta}\sec\theta\, d\theta$

$I = \displaystyle\int \frac{\cos^2\theta}{\sin^2\theta}\,\frac{1}{\cos\theta}\, d\theta$

$I = \displaystyle\int \csc\theta\cot\theta\, d\theta$

$I = -\csc\theta + C$

$I = \frac{-1}{\sin\theta} + C$



$\therefore I = -\frac{x}{\sqrt{x^2-1}} + C$ #

which is also given by

$I = -\frac{1}{2}\operatorname{sech}^{-1}\left(\frac{|x|}{2}\right) + C, \quad 0<|x|<2$

where $\displaystyle\int \frac{1}{u\sqrt{a^2-u^2}}\,du = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{|u|}{a}\right) + C, \quad 0<|u|<a$

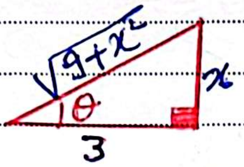
④ $\displaystyle\int \frac{1}{x\sqrt{9+x^2}}\,dx$

let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$

$$I = \int \frac{\cancel{3}\sec^{\cancel{2}}\theta}{\cancel{3}\tan\theta\, 3\sec\theta}\, d\theta \qquad \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$I = \frac{1}{3}\int \csc\theta\, d\theta$$

$$I = \frac{1}{3}\ln|\csc\theta - \cot\theta| + C$$



$$I = \frac{1}{3}\ln\left|\frac{\sqrt{9+x^2}}{x} - \frac{3}{x}\right| + C$$

Note that (Another Answer)

$$\int \frac{1}{|u|\sqrt{a^2+u^2}}\,du = \frac{-1}{a}\operatorname{csch}^{-1}\left(\frac{|u|}{a}\right) + C$$

i.e. $I = \frac{-1}{3}\operatorname{csch}^{-1}\left(\frac{|x|}{3}\right) + C$

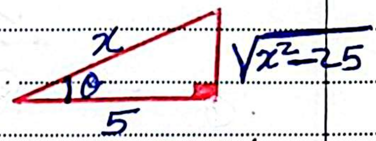
⑤ $\displaystyle\int \frac{1}{x^2\sqrt{x^2-25}}\,dx$, $|x| > 5$

let $x = 5\sec\theta$, $dx = 5\sec\theta\tan\theta\, d\theta$

$$I = \int \frac{5\sec\theta\tan\theta}{25\sec^2\theta\, 5\tan\theta}\, d\theta$$

$$I = \frac{1}{25}\int \cos\theta\, d\theta$$

$$I = \frac{1}{25}\sin\theta + C$$



$$I = \frac{1}{25}\left(\frac{\sqrt{x^2-25}}{x}\right) + C$$

Note that

$$\int \frac{1}{x\sqrt{x^2-25}}\,dx = \frac{1}{5}\sec^{-1}\left(\frac{x}{5}\right) + C \quad |x| > 5$$

which differs than our pb.

HW Evaluate the following Integrals

Remember

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

① $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

Hint: $x = 2\tan\theta$, ...

Ans: $I = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$

or $I = \sinh^{-1}\left(\frac{x}{2}\right) + C$

② $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

Hint: use $x = 3\sin\theta$, ...

Ans:

$I = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{9-x^2} + C$

③ $\displaystyle\int \frac{1}{\sqrt{25x^2-4}}$, $x > \frac{2}{5}$

Hint use $x = \frac{2}{5}\sec\theta$, ...

Ans: $I = \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$

④ $\displaystyle\int \frac{1}{x^4\sqrt{x^2-3}}\,dx$

Hint: use $x = \sqrt{3}\sec\theta$, ...

$I = \frac{1}{9}\left[\frac{\sqrt{x^2-3}}{x} - \frac{1}{3}\left(\frac{\sqrt{x^2-3}}{x}\right)^3\right] + C$

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