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Lecture (15) II Trigonometric Integrals

Evaluate the following Integrals (Integrals of powers of $\sin x$ and $\cos x$).

① $\int \sin^3 x\, dx$

$I = \int \sin^3 x\, dx$

$I = \int \sin^2 x \sin x\, dx$

$I = \int (1-\cos^2 x)\sin x\, dx$, $\sin^2 x = 1-\cos^2 x$

let $u = \cos x$, $du = -\sin x\, dx$

$I = -\int (1-u^2)\, du$

$I = -(u - \frac{u^3}{3}) + C$

$I = -u + \frac{1}{3}u^3 + C$

$\therefore I = -\cos x + \frac{1}{3}\cos^3 x + C$

② $\int \cos^2 x\, dx$

$I = \int \cos^2 x\, dx$

$I = \frac{1}{2}\int (1+\cos 2x)\, dx$,

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$I = \frac{1}{2}(x + \frac{\sin 2x}{2}) + C$

$\therefore I = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

• product of powers of sines and cosines

$$\int \sin^m x \cos^n x\, dx$$

m is odd: use $\sin^2 x = 1-\cos^2 x$, $u = \cos x$, $du = -\sin x\, dx$

n is odd: use, $\cos^2 x = 1-\sin^2 x$, $u = \sin x$, $du = \cos x\, dx$

m and n are even: use, $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$

③ $\int \sin^5 x \cos^2 x\, dx$

$I = \int \sin^4 x \cos^2 x \sin x\, dx$

$I = -\int (1-\cos^2 x)^2 \cos^2 x (-\sin x\, dx)$

$I = -\int (1-u^2)^2 u^2\, du$, $u = \cos x$

$I = -\int (1-2u^2+u^4)u^2\, du$

$I = -\int (u^2 - 2u^4 + u^6)\, du$

$I = -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{u^7}{7} + C$

$\therefore I = -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{\cos^7 x}{7} + C$

④ $\int \sin^2 x \cos^2 x\, dx$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$,

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$I = \int \sin^2 x \cos^2 x\, dx$

$I = \frac{1}{4}\int (1-\cos 2x)(1+\cos 2x)\, dx$

$I = \frac{1}{4}\int (1-\cos^2 2x)\, dx$

$I = \frac{1}{4}\int \sin^2 2x\, dx$

$I = \frac{1}{4}\cdot\frac{1}{2}\int (1-\cos 4x)\, dx$

$\therefore I = \frac{1}{8}(x - \frac{\sin 4x}{4}) + C$

Integrals of powers of tan x and sec x

⑤ $\int \tan^4 x\, dx$

$\tan^2 x = \sec^2 x - 1$

$I = \int \tan^2 x \tan^2 x\, dx$

$I = \int \tan^2 x(\sec^2 x - 1)\, dx$

$I = \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$

$I = \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx$

$I = \int \underline{\tan^2 x}\, \underline{\sec^2 x\, dx} - \int \sec^2 x\, dx + \int dx$

$\therefore I = \frac{1}{3}\tan^3 x - \tan x + x + C$

⑥ $\int \sec^3 x\, dx$

use Integration by parts

$I = \int \underbrace{\sec x}_{u}\, \underbrace{\sec^2 x\, dx}_{dv}$

$u = \sec x \qquad dv = \sec^2 x\, dx$

$du = \sec x \tan x \qquad v = \tan x$

$I = \sec x \tan x - \int \sec x \tan^2 x\, dx$

$I = \sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx$

$I = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$\therefore I = \sec x \tan x - I + \ln|\sec x + \tan x| + C$

$\therefore 2I = \sec x \tan x + \ln|\sec x + \tan x| + C$

$\therefore I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$