Lecture (14)

Techniques of Integration

(I) Integration by parts

Theorem If $u = f(x)$ and $v = g(x)$, and if f' and g' are continuous then $\int u\,dv = uv - \int v\,du$

proof

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow \int d(uv) = \int u\,dv + \int v\,du$$

$$\therefore \int u\,dv = uv - \int v\,du$$

* Evaluate the following Integrals

① $\int \underbrace{x}_{u}\, \underbrace{e^{2x}dx}_{dv}$

$u = x$, $dv = e^{2x}dx$

$du = dx$, $v = \frac{e^{2x}}{2}$

$I = \int x e^{2x} dx$

$I = \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$

$I = \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{e^{2x}}{2}\right) + C$

$\therefore I = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

② $\int_0^{\pi/3} \underbrace{x}_{u}\, \underbrace{\sec^2 x\, dx}_{dv}$

$u = x$, $dv = \sec^2 x\, dx$

$du = dx$, $v = \tan x$

let $I = \int_0^{\pi/3} x \sec^2 x\, dx$

$\therefore I = [x \tan x]_0^{\pi/3} - \int_0^{\pi/3} \tan x\, dx$

$I = [x \tan x + \ln|\cos x|]_0^{\pi/3}$

$I = \left[\frac{\pi}{3}\sqrt{3} + \ln(1/2)\right] - [0 + \cancel{\ln 1}_{0}]$

$\therefore I = \frac{\pi}{3}\sqrt{3} + \ln(1/2) \simeq 1.12$

③ $\int \underbrace{x}_{u}\, \underbrace{\sec x \tan x\, dx}_{dv}$

$u = x$, $dv = \sec x \tan x\, dx$

$du = dx$, $v = \sec x$

$I = \int x \sec x \tan x\, dx$

$I = x \sec x - \int \sec x\, dx$

$I = x \sec x - \ln|\sec x + \tan x| + C$

P(14)

④ $\int \underbrace{\tan^{-1}x}_{u}\, \underbrace{dx}_{dv}$

$u = \tan^{-1}x, \quad dv = dx$

$du = \frac{1}{1+x^2}dx, \quad v = x$

$I = \int \tan^{-1}x\, dx$

$I = x\tan^{-1}x - \int \frac{x}{1+x^2}dx$

$I = x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$

$\therefore I = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$

⑤ $\int \underbrace{x}_{u}\, \underbrace{\csc^2 x\, dx}_{dv}$

$u = x, \quad dv = \csc^2 x\, dx$

$du = dx, \quad v = -\cot x$

$I = \int x\csc^2 x\, dx$

$I = -x\cot x + \int \cot x\, dx$

$\therefore I = -x\cot x + \ln|\sin x| + C$

⑥ $\int \underbrace{(\ln x)^2}_{u}\, \underbrace{dx}_{dv}$

$u = (\ln x)^2, \quad dv = dx$

$du = 2(\ln x)\left(\frac{1}{x}\right)dx, \quad v = x$

$I = x(\ln x)^2 - 2\int \ln x\, dx$ ---- ①

Apply again Integration by parts to get

$\int \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv}$

$u = \ln x, \quad dv = dx$

$du = \frac{1}{x}dx, \quad v = x$

$\int \ln x\, dx$

$= x\ln x - \int dx$

$= x\ln x - x + C$ ②

Substitute ② in ①, We get

$I = x(\ln x)^2 - 2(x\ln x - x) + C$ #

Some Important Remarks

(1) You can choose u in this order

LIATE → Exponential

logs, Inverse, Algebraic, Trig.

(2) You can Solve Examples ①, ②, ③ and ⑤ by Tabular Method (DI Method) [D = Derivative, I = Integral] in which a polynomial factor such as $x, x^2, x^3, \ldots$ is multiplied by a fn like e^x, $\sin x$, $\cos x$ ... .

e.g. To find $\int x e^{2x} dx$ ... See EX①

sign	D	I
+	x	e^{2x}
−	1	$\frac{e^{2x}}{2}$
+	0	$\frac{1}{4}e^{2x}$

$I = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

* Use Tabular Method to find

$\int x^2 e^{-2x} dx$

sign	D	I
+	x^2	e^{-2x}
−	$2x$	$\frac{e^{-2x}}{-2}$
+	2	$\frac{e^{-2x}}{4}$
−	0	$\frac{e^{-2x}}{-8}$

$I = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$ #

HW Solve again by using Integration by parts.