Lecture (13) Inverse Hyperbolic Functions

Theorem

(1) $\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$

(2) $\cosh^{-1}x = \ln(x+\sqrt{x^2-1})$, $x \geq 1$

(3) $\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$, $|x|<1$

(4) $\coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$, $|x|>1$

(5) $\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$, $0<x\leq 1$

(6) $\operatorname{csch}^{-1}x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$, $x \neq 0$

Theorem

Derivative of Inv. Hyp. Fns	Integration of Inv. Hyp. Fns
(1) $\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$	$\int \frac{1}{\sqrt{a^2+u^2}}\,du = \sinh^{-1}\left(\frac{u}{a}\right)+C$, $a>0$
(2) $\frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$, $u>1$	$\int \frac{1}{\sqrt{u^2-a^2}}\,du = \cosh^{-1}\left(\frac{u}{a}\right)+C$, $0<a<u$
(3) $\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}$, $\lvert u\rvert<1$	$\int \frac{1}{a^2-u^2}\,du = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)+C$, $\lvert u\rvert<a$
(4) $\frac{d}{dx}\coth^{-1}u = \frac{-1}{1-u^2}\frac{du}{dx}$, $\lvert u\rvert>1$	$\int \frac{1}{a^2-u^2}\,du = \frac{-1}{a}\coth^{-1}\left(\frac{u}{a}\right)+C$, $\lvert u\rvert>a$
(5) $\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$, $0<u<1$	$\int \frac{1}{u\sqrt{a^2-u^2}}\,du = \frac{-1}{a}\operatorname{sech}^{-1}\left(\frac{\lvert u\rvert}{a}\right)+C$, $0<\lvert u\rvert<a$
(6) $\frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-1}{\lvert u\rvert\sqrt{1+u^2}}\frac{du}{dx}$, $u\neq 0$	$\int \frac{1}{\lvert u\rvert\sqrt{a^2+u^2}}\,du = \frac{-1}{a}\operatorname{csch}^{-1}\left(\frac{\lvert u\rvert}{a}\right)+C$, $u\neq 0$

where $u=g(x)$, and g is differentiable function.

EX Find $f'(x)$ or $\frac{dy}{dx}$ ...

① $f(x) = \ln(\cosh^{-1} 4x)$

Ans:

$$f'(x) = \frac{1}{\cosh^{-1}(4x)} \frac{d}{dx}(\cosh^{-1} 4x)$$

$$f'(x) = \frac{1}{\cosh^{-1} 4x} \cdot \frac{1}{\sqrt{16x^2-1}} (4)$$

$$f'(x) = \frac{4}{\sqrt{16x^2-1}\ \cosh^{-1} 4x}$$

② $f(x) = \sinh^{-1}(3x-1)$

Ans:

$$f'(x) = \frac{1}{\sqrt{1+(3x-1)^2}} (3)$$

$$f'(x) = \frac{3}{\sqrt{1+(3x-1)^2}}$$

③ $f(x) = \tanh^{-1}(x+1)$

Ans: $f'(x) = \frac{1}{1-(x+1)^2} \frac{d}{dx}(x+1)$

$$\therefore f'(x) = \frac{1}{-x^2-2x} = \frac{-1}{x(x+3)}$$

④ $y = \sinh^{-1}(\tan x)$

Ans: $\frac{dy}{dx} = |\sec x|$ HW

EX Find the following Integrals

① $\int \frac{1}{\sqrt{81+16x^2}} dx$

$$I = \frac{1}{4}\int \frac{4}{\sqrt{9^2+(4x)^2}} dx$$

$$\therefore I = \frac{1}{4} \sinh^{-1}\left(\frac{4x}{9}\right) + C$$

② $\int \frac{1}{49-4x^2} dx$

$u = 2x$, $du = 2dx$ $\Rightarrow dx = \frac{1}{2} du$

$$I = \frac{1}{2}\int \frac{1}{7^2-u^2} du$$

$$I = \frac{1}{2} \cdot \frac{1}{7} \tanh^{-1}\left(\frac{u}{7}\right) + C$$

$$\therefore I = \frac{1}{14} \tan^{-1}\left(\frac{2x}{7}\right) + C$$

③ $\int \frac{e^x}{\sqrt{e^{2x}-16}} dx$

$u = e^x$, $du = e^x dx$

$$I = \int \frac{1}{\sqrt{u^2-4^2}} du$$

$$I = \cosh^{-1}\left(\frac{u}{4}\right) + C$$

$$\therefore I = \cosh^{-1}\left(\frac{e^x}{4}\right) + C$$

④ $\int \frac{1}{x\sqrt{9-x^4}} dx$ $\quad \times 2x$ / $\times 2x$

$$I = \frac{1}{2}\int \frac{2x}{x^2\sqrt{9-(x^2)^2}} dx$$

$$\therefore I = \frac{-1}{6} \operatorname{sech}^{-1}\left(\frac{x^2}{3}\right) + C$$

⑤ $\int \frac{1}{\sqrt{x}\sqrt{4+x}} dx$ HW

Ans: $I = 2\sinh^{-1}\left(\frac{\sqrt{x}}{2}\right) + C$

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