Lecture (12) Hyperbolic Functions

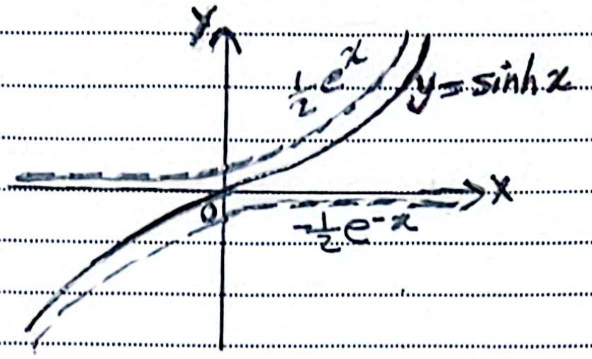
The Hyperbolic Functions are defined as follows

(1) $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$\sinh 0 = 0$

Domain $= \mathbb{R} = (-\infty, \infty)$

Range $= \mathbb{R} = (-\infty, \infty)$



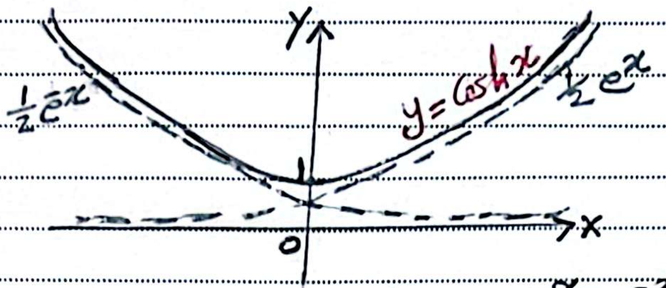
(2) $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

$\cosh 0 = 1$

Domain $= \mathbb{R}$

Range $= [1, \infty)$

It's looks like a catena منحنى الكتينة Curve (not a parabola).



(3) $\tanh x = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

(4) $\coth x = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}, \quad x \neq 0$

(5) $\operatorname{sech} x = \dfrac{2}{e^{x} + e^{-x}}$

(6) $\operatorname{csch} x = \dfrac{2}{e^{x} - e^{-x}}$

- Theorem (Identities for Hyperbolic Functions).

(1) $\cosh^2 x - \sinh^2 x = 1$

(2) $1 - \tanh^2 x = \operatorname{sech}^2 x$ $\quad \div \cosh^2 x$

(3) $\coth^2 x - 1 = \operatorname{csch}^2 x$ $\quad \div \sinh^2 x$

- Theorem

Differentiation	Integration
(1) $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$	$\int \cosh u\, du = \sinh u + C$
(2) $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$	$\int \sinh u\, du = \cosh u + C$
(3) $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$	$\int \operatorname{sech}^2 u\, du = \tanh u + C$
(4) $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$	$\int \operatorname{csch}^2 u\, du = -\coth u + C$
(5) $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$	$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$
(6) $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$	$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$

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EX① find $f'(x)$ for each of the following

(1) $f(x) = \cosh\sqrt{4x^2+3}$

$$f'(x) = \sinh\sqrt{4x^2+3}\ \frac{1}{2\sqrt{4x^2+3}}(8x)$$

$$f'(x) = \frac{4x\sinh\sqrt{4x^2+3}}{\sqrt{4x^2+3}}$$

HW ③ $f(t) = \tanh\sqrt{1+t^2}$

$$\frac{d}{dx}\sqrt{u} = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

Ans:

$$f'(t) = \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

(2) $f(x) = x^{\cosh x}$ Take ln

$$\ln f(x) = \cosh x \ln x$$

Diff. w.r.t x

$$\frac{1}{f(x)}f'(x) = \cosh x\,\frac{1}{x} + \sinh x\ln x$$

therefore,

$$f'(x) = x^{\cosh x}\left[\frac{1}{x}\cosh x + \sinh x\ln x\right]$$

EX② Evaluate the following Integrals

(1) $\int x^2\cosh(x^3)\,dx$

$$I = \frac{1}{3}\int \cosh(x^3)\,3x^2\,dx$$

$$= \frac{1}{3}\sinh(x^3) + C$$

(2) $\int \frac{1}{\cosh^2(3x)}\,dx$

$$I = \frac{1}{3}\int \operatorname{sech}^2 3x\ 3\,dx$$

$$I = \frac{1}{3}\tanh 3x + C$$

(3) $\int \frac{\sinh x}{1+\sinh^2 x}\,dx$

$\cosh^2 x - \sinh^2 x = 1$

$$I = \int \frac{\sinh x}{\cosh^2 x}\,dx$$

$$I = \int \operatorname{sech} x\tanh x\,dx$$

$$I = -\operatorname{sech} x + C$$

(4) $\int \coth 5x\,dx$ HW

$$I = \int \frac{\cosh 5x}{\sinh 5x}\,dx$$

$u = \sinh 5x$

Ans:

$$I = \frac{1}{5}\ln|\sinh 5x| + C$$

(5) $\int_0^1 \sinh^2 x\,dx$

$$I = \frac{1}{2}\int_0^1(\cosh 2x - 1)\,dx$$

$$I = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$

$$I = \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$

* Note that; the following identities
- $\sinh 2x = 2\sinh x\cosh x$
- $\cosh 2x = \cosh^2 x + \sinh^2 x$
- $\cosh^2 x = \frac{\cosh 2x + 1}{2}$
- $\sinh^2 x = \frac{\cosh 2x - 1}{2}$