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Lecture (11) — Differentiating and Integrating Inverse Trigonometric Functions

* Derivative of Inv. Trig. functions (arc sin u, arc cos u, ...) ($\sin^{-1}u$, $\cos^{-1}u$)

- $\frac{d}{dx}(\sin^{-1}u)=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, $|u|<1$

$\frac{d}{dx}(\cos^{-1}u)=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$, $|u|<1$

- $\frac{d}{dx}(\tan^{-1}u)=\frac{1}{1+u^2}\frac{du}{dx}$

$\frac{d}{dx}(\cot^{-1}u)=\frac{-1}{1+u^2}\frac{du}{dx}$

- $\frac{d}{dx}(\sec^{-1}u)=\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$, $|u|>1$

$\frac{d}{dx}(\csc^{-1}u)=-\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}$, $|u|>1$

when $u=g(x)$, g is a differentiable function.

* Integrals of Inv. Trig. functions

$$\int\frac{1}{\sqrt{a^2-u^2}}du=\sin^{-1}\left(\frac{u}{a}\right)+c\ ,\quad |u|<a$$

$$\int\frac{1}{a^2+u^2}du=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$$

$$\int\frac{1}{u\sqrt{u^2-a^2}}du=\frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)+c\ ,\quad |u|>a$$

Examples

(I) find $f'(x)$ for each of the following

① $f(x)=e^{\cos^{-1}(4x+1)}$

Ans: $f'(x)=e^{\cos^{-1}(4x+1)}\cdot\frac{-1}{\sqrt{1-(4x+1)^2}}(4)$ ($u=4x+1$)

$\therefore f'(x)=\frac{-4}{\sqrt{1-(4x+1)^2}}e^{\cos^{-1}(4x+1)}$

② $f(x)=\sin^{-1}\sqrt{x}$

$f'(x)=\frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right)$ ($u=\sqrt{x}$)

$\therefore f'(x)=\frac{1}{2\sqrt{x}\sqrt{1-x}}$

③ $f(x) = \tan^{-1}(2x^2+3)$

$u = 2x^2+3$

$$f'(x) = \frac{1}{1+(2x^2+3)^2} \cdot 4x$$
$$= \frac{4x}{1+(2x^2+3)^2}$$

④ $f(x) = \sec^{-1}(3+\sin 3x)$

$$f'(x) = \frac{1}{(3+\sin 3x)\sqrt{(3+\sin 3x)^2-1}} \cdot (\cos 3x)(3)$$

$u = 3+\sin 3x$

$$f'(x) = \frac{3\cos 3x}{(3+\sin 3x)\sqrt{(3+\sin 3x)^2-1}}$$

⑤ $f(x) = \ln\left|e^{5x} + \sec^{-1}(3x)\right|$

Ans: $f'(x) = \dfrac{5e^{5x} + \dfrac{3}{3x\sqrt{9x^2-1}}}{e^{5x}+\sec^{-1}(3x)}$

(II) Find the Following Integrals:

① $\displaystyle\int \frac{x^2}{9+x^6}\,dx$

$$I = \int \frac{x^2}{9+x^6}\,dx$$

$u = x^3$
$du = 3x^2\,dx$

$$I = \frac{1}{3}\int \frac{1}{3^2+u^2}\,du$$
$$I = \frac{1}{3}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$
$$\therefore I = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C$$

② $\displaystyle\int \frac{x}{\sqrt{16-x^4}}\,dx$

$$I = \int \frac{x}{\sqrt{16-x^4}}\,dx$$

$u = x^2$
$du = 2x\,dx$

$$I = \frac{1}{2}\int \frac{1}{\sqrt{4^2-u^2}}\,du$$
$$I = \frac{1}{2}\sin^{-1}\left(\frac{u}{4}\right) + C$$
$$\therefore I = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{4}\right) + C$$

③ $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$

$\displaystyle\int \frac{1}{\sqrt{u}}\,du = 2\sqrt{u}+C$

Ans: $I = -\sqrt{16-x^2} + C$

④ $\displaystyle\int \frac{e^{2x}}{25+e^{4x}}\,dx$

$u = e^{2x}$

Ans: $I = \dfrac{1}{10}\tan^{-1}\left(\dfrac{e^{2x}}{5}\right) + C$

⑤ $\displaystyle\int \frac{1}{x\sqrt{1-(\ln x)^2}}\,dx$

Ans: $I = \sin^{-1}(\ln x) + C$

$u = \ln x$

⑥ $\displaystyle\int \frac{1}{(x-1)\sqrt{x^2-2x-3}}\,dx$

$$I = \int \frac{1}{(x-1)\sqrt{(x-1)^2-4}}\,dx$$

. . . $u = x-1$

$$I = \frac{1}{2}\sec^{-1}\left(\frac{x-1}{2}\right) + C$$