Lecture (10)

- General Exponential Function and Logarithmic Function

$\because e^{\ln x} = x \;\; \forall x > 0 \Rightarrow \therefore e^{\ln a} = a, \quad a > 0$

So, we can define ... a^x as follows.

- Definition of a^x, the exponential function with base a.

$$a^x = (e^{\ln a})^x$$

i.e. $a^x = e^{x \ln a}, \quad a > 0, \quad x \in \mathbb{R}$

Note: at $a = e$ $\Rightarrow a^x = e^x$

- Rules for the exponents of a^x

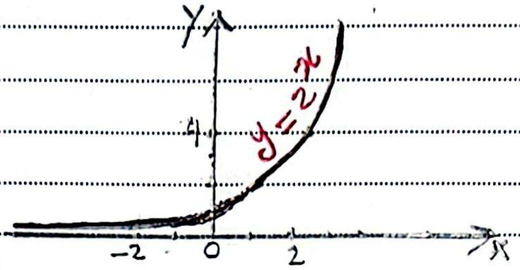
Let $a > 0$ and $b > 0$. If u and v are real numbers, then

$a^u a^v = a^{u+v}, \quad (a^u)^v = a^{uv}, \quad (ab)^u = a^u b^u$

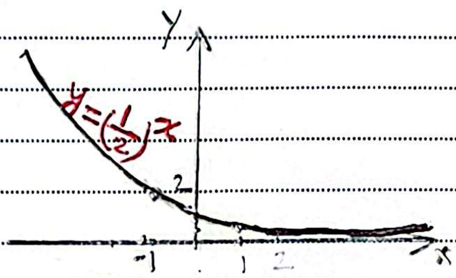
$\frac{a^u}{a^v} = a^{u-v}, \quad \left(\frac{a}{b}\right)^u = \frac{a^u}{b^u}.$

* Note that

(i) $y = a^x, \quad a > 1$ is increasing on $(-\infty, \infty)$



(ii) $y = a^x, \quad 0 < a < 1$ is decreasing on $(-\infty, \infty)$.



- Theorem

Derivative of exp. fn with base a

(i) $\frac{d}{dx}(a^x) = a^x \ln a$

(ii) $\frac{d}{dx}(a^u) = (a^u \ln a)\frac{du}{dx}$

- Theorem

Integral of exponential fn with base a.

(i) $\int a^x dx = \left(\frac{1}{\ln a}\right) a^x + c$

(ii) $\int a^u du = \left(\frac{1}{\ln a}\right) a^u + c$

* Note that $u = g(x)$, g is differentiable function.

- **Logarithmic Function with base a**

Definition (1) $\log_a x$ (the logarithm of x with base a)

$$y = \log_a x \quad \text{iff} \quad x = a^y$$

Definition (2)

$$\log_a x = \frac{\ln x}{\ln a}$$

$y = \log_a x$
$\Rightarrow a^y = x$
$\Rightarrow y \ln a = \ln x$
$\Rightarrow y = \frac{\ln x}{\ln a}$

Note: at $a = e$ $\Rightarrow \log_a x = \ln x$

Note that $\log_{10} x$ is denoted by $\log x$ (common logarithm)

Theorem

(i) $$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{1}{x}$$

(ii) $$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx},$$

where u is a positive differentiable function of x.

- **Examples**

① Find $f'(x)$ if $f(x) = \log(\ln x)$

Ans:

$$f'(x) = \frac{1}{\ln 10} \cdot \frac{1}{\ln x} \frac{d}{dx}(\ln x)$$

$$= \frac{1}{\ln 10} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

② Find $f'(x)$ if $f(x)$ is

(i) π^π (ii) x^4 (iii) x^π (iv) π^x
(v) x^{2x}

Ans:

(i) $f(x) = \pi^\pi$ (Constant) $\Rightarrow f'(x) = 0$

(ii) $f(x) = x^4 \Rightarrow f'(x) = 4x^3$

(iii) $f(x) = x^\pi \Rightarrow f'(x) = \pi x^{\pi - 1}$

(iv) $f(x) = x^{2x}$

$\xrightarrow{\ln} \ln f(x) = \ln(x^{2x})$

$\ln f(x) = 2x \ln x$

Diff. w.r.t x

$\Rightarrow \frac{1}{f(x)} f'(x) = 2x\left(\frac{1}{x}\right) + 2\ln x$

$\therefore f'(x) = 2x^{2x}(1 + \ln x)$. #

③ Find $\frac{d}{dx}(10^{\sin x})$

Ans:

$$\frac{d}{dx}(10^{\sin x})$$

$$= 10^{\sin x} \ln 10 \frac{d}{dx} \sin x$$

$$= (10^{\sin x} \ln 10) \cos x$$

$$= \ln 10 \cdot 10^{\sin x} \cos x.$$

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

④ Evaluate $\int x \, 3^{x^2} dx$

Ans:

$$I = \int 3^{x^2} x \, dx$$

let $u = x^2$

$\Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2} du$

$$I = \frac{1}{2}\int 3^u du$$

$$I = \frac{1}{2}\left(\frac{1}{\ln 3}\right) 3^u + C$$

$$\therefore I = \left(\frac{1}{2\ln 3}\right) 3^{x^2} + C \quad \#$$

⑤ Evaluate

(i) $\int_{-1}^{0} 3^x \, dx$ (ii) $\int \frac{5^{\tan x}}{\cos^2 x} dx$

Ans:

(i) $\int_{-1}^{0} 3^x dx = \left[\frac{3^x}{\ln 3}\right]_{-1}^{0}$

$= \frac{1}{\ln 3}\left(1 - \frac{1}{3}\right)$

$= \frac{2}{3\ln 3} \approx 0.61$

(ii) $I = \int \frac{5^{\tan x}}{\cos^2 x} dx$

let $u = \tan x \Rightarrow du = \sec^2 x \, dx$

i.e. $du = \frac{1}{\cos^2 x} dx$

$I = \int 5^u du$

$I = \frac{5^u}{\ln 5} + C$

$I = \frac{1}{\ln 5} 5^{\tan x} + C$

⑦ Find $\int \frac{e^x}{1+e^x} dx$ HW

Ans: $I = \ln(1+e^x) + C$

⑧ $\int \frac{1}{1+e^x} dx$

$I = \int \frac{1}{1+e^x} dx$

$I = \int \left[\frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x}\right] dx$

$I = \int \left(1 - \frac{e^x}{1+e^x}\right) dx$

$\therefore I = x - \ln(1+e^x) + C$

⑥ Evaluate

$$\int \frac{(1+3^{-x})^{2024}}{3^x} dx$$

Ans:

let $u = 1 + 3^{-x}$

$\Rightarrow du = 3^{-x} \ln 3 (-1) dx$

$\Rightarrow du = -\frac{\ln 3}{3^x} dx$

$I = \int (1+3^{-x})^{2024} \frac{dx}{3^x}$

$I = \frac{-1}{\ln 3} \int u^{2024} du$

$I = \frac{-1}{\ln 3}\left(\frac{u^{2025}}{2025}\right) + C$ power rule

$\therefore I = \frac{-(1+3^{-x})^{2025}}{2025 \ln 3} + C$

⑨ Find $\int_0^1 2^{-\theta} d\theta$ HW

Hint $u = -\theta$ ...

Ans: $I = \frac{1}{2\ln 2}$

⑩ Find each of the following HW

(a) $\frac{d}{dx} \log_{10}(3x+1)$

(b) $\int \frac{\log_2 x}{x} dx$

(c) $\int \frac{\tan(e^{-3x})}{e^{3x}} dx$

Ans: (a) $\frac{3}{(\ln 10)(3x+1)}$

(b) $(\ln x)^2 / 2\ln 2 + C$

(c) $\frac{1}{3} \ln |\cos(e^{-3x})| + C$