Riemann Integrability and the Lebesgue Integral

1 Riemann Integrability and Lebesgue Integrability

Throughout these notes, m denotes Lebesgue measure on $\mathbb{R}$. When we write

$$\int_E f\,dm,$$

we mean the Lebesgue integral of f over the measurable set E. When we write

$$\int_a^b f(x)\,dx,$$

we mean the usual Riemann integral, provided it exists.

1.1 The Lebesgue Criterion for Riemann Integrability

Definition 1.1 (Set of measure zero)**.** A set $E \subseteq \mathbb{R}$ has *measure zero* if, for every $\varepsilon > 0$, there exist open intervals $I_1, I_2, I_3, \ldots$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} |I_n| < \varepsilon.$$

Here $|I_n|$ denotes the length of the interval I_n.

Definition 1.2 (Continuous almost everywhere)**.** A function $f : [a,b] \to \mathbb{R}$ is *continuous almost everywhere* on $[a,b]$ if the set of points where f is discontinuous has measure zero.

Lebesgue Criterion for Riemann Integrability

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ if and only if the set of discontinuities of f has measure zero.
Equivalently:

A bounded function is Riemann integrable exactly when it is continuous almost everywhere.

In this case, f is Lebesgue integrable and the two integrals agree:

$$\boxed{\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm.}$$

Remark 1.3. This result is often called *Lebesgue's criterion for Riemann integrability.* It should not be confused with the Riemann–Lebesgue lemma from Fourier analysis.

1.2 Immediate Consequences

Consequences

If f is Riemann integrable on $[a,b]$, then:

- f is measurable,
- f is Lebesgue integrable,
- and the Riemann and Lebesgue integrals are equal:

$$\int_a^b f(x)\,dx = \int_{[a,b]} f\,dm.$$

Therefore, for Riemann integrable functions, it is common to use the notation

$$\int_a^b f(x)\,dx$$

for both the Riemann integral and the Lebesgue integral over $[a,b]$, because the two numbers are the same.

Main Conclusion

Every Riemann integrable function on a compact interval is Lebesgue integrable. However, the converse is false: there are functions that are Lebesgue integrable but not Riemann integrable.

1.3 Example: Lebesgue Integrable but Not Riemann Integrable

Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0, & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

This is the *Dirichlet function*, also written as $f = \mathbf{1}_{\mathbb{Q}\cap[0,1]}$.

Step 1: The Function is Discontinuous Everywhere

Fix any point $c \in [0,1]$. Every interval around c contains rational numbers and irrational numbers. Therefore, there are sequences (r_n) and (s_n) such that

$$r_n \in \mathbb{Q} \cap [0,1], \qquad s_n \in [0,1] \setminus \mathbb{Q}, \qquad r_n \to c, \qquad s_n \to c.$$

For these sequences,

$$f(r_n) = 1 \qquad \text{and} \qquad f(s_n) = 0.$$

If f were continuous at c, then both $f(r_n)$ and $f(s_n)$ would have to converge to $f(c)$. But the two limits are different:

$$\lim_{n\to\infty} f(r_n) = 1, \qquad \lim_{n\to\infty} f(s_n) = 0.$$

Hence f is not continuous at c. Since c was arbitrary, f is discontinuous at every point of $[0,1]$.

Step 2: Not Riemann Integrable

The set of discontinuities of f is all of $[0,1]$. Since

$$m([0,1]) = 1 \neq 0,$$

the set of discontinuities does not have measure zero. By the Lebesgue criterion, f is not Riemann integrable.

We can also see this directly using upper and lower sums. Let P be any partition of $[0,1]$. On every subinterval of P, the function takes both values 0 and 1. Therefore, on every subinterval,

$$\inf f = 0 \qquad \text{and} \qquad \sup f = 1.$$

So every lower sum is

$$L(f,P) = 0,$$

and every upper sum is

$$U(f,P) = 1.$$

Thus the lower Riemann integral is 0, while the upper Riemann integral is 1. Since these are not equal, f is not Riemann integrable.

Step 3: Lebesgue Integrable

The rational numbers are countable, so $\mathbb{Q} \cap [0,1]$ has measure zero. Therefore,

$$f = 1 \quad \text{only on a measure-zero set,}$$

and

$$f = 0 \quad \text{almost everywhere on } [0,1].$$

Hence

$$\int_{[0,1]} f\,dm = \int_{\mathbb{Q}\cap[0,1]} 1\,dm + \int_{[0,1]\setminus\mathbb{Q}} 0\,dm = 0 + 0 = 0.$$

Thus

$$\boxed{\int_{[0,1]} f\,dm = 0.}$$

1.4 Example: Using the Lebesgue Criterion

Determine whether

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1, \\ 0, & x = 1, \end{cases}$$

is Riemann integrable on $[0, 2]$. If it is, compute

$$\int_0^2 f(x)\,dx.$$

Solution. For $x \neq 1$, factor the numerator:

$$x^2 - 1 = (x-1)(x+1).$$

Therefore,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1, \qquad x \neq 1.$$

So f agrees with the continuous function $g(x) = x + 1$ at every point except possibly $x = 1$.

At $x = 1$, we have

$$f(1) = 0.$$

But

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1) = 2.$$

Since

$$f(1) = 0 \neq 2,$$

f is discontinuous at $x = 1$.

If $x \neq 1$, then near x, the formula is $f(x) = x + 1$, so f is continuous there. Hence the only point of discontinuity is

$$\{1\}.$$

A single point has measure zero. Also, f is bounded on $[0, 2]$. Indeed,

$$f(1) = 0, \qquad f(x) = x + 1 \text{ for } x \neq 1,$$

and for $x \in [0, 2]$,

$$0 \leq x + 1 \leq 3.$$

Therefore f is bounded.

By the Lebesgue criterion, f is Riemann integrable on $[0, 2]$.

Changing a function at one point does not change its Riemann integral or its Lebesgue integral. Therefore,

$$\int_0^2 f(x)\,dx = \int_0^2 (x+1)\,dx.$$

Now compute:

$$\int_0^2 (x+1)\,dx = \int_0^2 x\,dx + \int_0^2 1\,dx.$$

We have

$$\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_0^2 = \frac{2^2}{2} - 0 = 2,$$

and

$$\int_0^2 1\,dx = [x]_0^2 = 2.$$

Thus

$$\int_0^2 (x+1)\,dx = 2+2=4.$$

Hence

$$\boxed{\int_0^2 f(x)\,dx = 4.}$$

□

2 Improper Integrals

Lebesgue Integral on Unbounded Sets

The Lebesgue integral can be defined on unbounded measurable sets such as $\mathbb{R}$, $[a,\infty)$, and $(0,\infty)$, provided the function is measurable.
However, the integral may be infinite. For example,

$$\int_{\mathbb{R}} 1\,dm = \infty.$$

So the function $f(x)=1$ is measurable on $\mathbb{R}$, but it is not in $L^1(\mathbb{R})$, because

$$\int_{\mathbb{R}} |1|\,dm = \infty.$$

Improper Riemann Integral

The Riemann integral is originally defined on bounded intervals. On an unbounded interval, it is extended by a limit. If the limit exists as a finite real number, we define

$$\int_a^\infty f(x)\,dx := \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

This is called an *improper Riemann integral*.

Similarly, for a singularity at the left endpoint, we define

$$\int_0^1 f(x)\,dx := \lim_{\varepsilon\downarrow 0}\int_\varepsilon^1 f(x)\,dx,$$

provided the limit exists as a finite real number.

2.1 Nonnegative Functions

Nonnegative Case

Suppose $f \geq 0$ on $[a, \infty)$, and suppose f is Riemann integrable on every compact interval $[a, b]$. If

$$\int_a^\infty f(x)\,dx$$

converges as an improper Riemann integral, then f is Lebesgue integrable on $[a, \infty)$, and

$$\boxed{\int_{[a,\infty)} f\,dm = \int_a^\infty f(x)\,dx.}$$

Proof. For each $n \in \mathbb{N}$ with $n \geq a$, define the truncated function

$$f_n(x) = f(x)\mathbf{1}_{[a,n]}(x).$$

Explicitly,

$$f_n(x) = \begin{cases} f(x), & a \leq x \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Since $[a, n] \subseteq [a, n+1]$, we have

$$0 \leq f_n(x) \leq f_{n+1}(x) \leq f(x) \qquad \text{for every } x.$$

Also, for every $x \in [a, \infty)$, once $n \geq x$, we have $f_n(x) = f(x)$. Therefore,

$$f_n(x) \uparrow f(x) \qquad \text{pointwise on } [a, \infty).$$

By the Monotone Convergence Theorem,

$$\int_{[a,\infty)} f\,dm = \lim_{n\to\infty} \int_{[a,\infty)} f_n\,dm.$$

But $f_n = 0$ outside $[a, n]$, so

$$\int_{[a,\infty)} f_n\,dm = \int_{[a,n]} f\,dm.$$

Since f is Riemann integrable on $[a, n]$, the Riemann and Lebesgue integrals agree on $[a, n]$. Hence

$$\int_{[a,n]} f\,dm = \int_a^n f(x)\,dx.$$

Therefore,

$$\int_{[a,\infty)} f\,dm = \lim_{n\to\infty} \int_a^n f(x)\,dx = \int_a^\infty f(x)\,dx.$$

Since the improper integral is finite by assumption, the Lebesgue integral is finite. Thus $f \in L^1([a, \infty))$. $\square$

2.2 Example: Computing $\int_0^\infty e^{-x}\,dx$

Solution. Define

$$f(x) = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then $f \geq 0$, and f is measurable.

For each $n \in \mathbb{N}$, define

$$f_n(x) = f(x)\mathbf{1}_{[0,n]}(x).$$

That is,

$$f_n(x) = \begin{cases} e^{-x}, & 0 \leq x \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

Since $[0,n] \subseteq [0,n+1]$, we have

$$0 \leq f_n \leq f_{n+1} \leq f.$$

Also,

$$f_n(x) \to f(x) \qquad \text{for every } x \in \mathbb{R}.$$

Therefore, by the Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f\,dm = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\,dm.$$

Because $f_n = 0$ outside $[0,n]$,

$$\int_{\mathbb{R}} f_n\,dm = \int_{[0,n]} e^{-x}\,dm = \int_0^n e^{-x}\,dx.$$

Now compute the ordinary Riemann integral:

$$\int_0^n e^{-x}\,dx = \left[-e^{-x}\right]_0^n = -e^{-n} - (-e^0) = 1 - e^{-n}.$$

Taking the limit gives

$$\int_0^\infty e^{-x}\,dx = \lim_{n\to\infty}(1 - e^{-n}).$$

Since $e^{-n} \to 0$,

$$\lim_{n\to\infty}(1 - e^{-n}) = 1.$$

Therefore,

$$\boxed{\int_0^\infty e^{-x}\,dx = 1.}$$

$\square$

2.3 Example: Computing $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Solution. Define

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & 0 < x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The function has a singularity at $x = 0$, so we approximate it from the right.

For $n = 2, 3, 4, \ldots$, define

$$f_n(x) = f(x)\mathbf{1}_{[1/n,1]}(x).$$

Equivalently,

$$f_n(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & \dfrac{1}{n} \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

As n increases, the interval $[1/n, 1]$ expands toward $(0, 1]$. Therefore,

$$0 \le f_n \le f_{n+1} \le f,$$

and

$$f_n(x) \to f(x) \qquad \text{for every } x \in \mathbb{R}.$$

By the Monotone Convergence Theorem,

$$\int_{\mathbb{R}} f\,dm = \lim_{n\to\infty} \int_{\mathbb{R}} f_n\,dm.$$

Since $f_n = 0$ outside $[1/n, 1]$,

$$\int_{\mathbb{R}} f_n\,dm = \int_{1/n}^{1} \frac{1}{\sqrt{x}}\,dx.$$

Rewrite the integrand:

$$\frac{1}{\sqrt{x}} = x^{-1/2}.$$

Then

$$\int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} = 2\sqrt{x}.$$

Therefore,

$$\int_{1/n}^{1} \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_{1/n}^{1} = 2\sqrt{1} - 2\sqrt{\frac{1}{n}} = 2 - \frac{2}{\sqrt{n}}.$$

Taking limits,

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{n\to\infty} \left(2 - \frac{2}{\sqrt{n}}\right).$$

Since $2/\sqrt{n} \to 0$,

$$\lim_{n\to\infty} \left(2 - \frac{2}{\sqrt{n}}\right) = 2.$$

Thus

$$\boxed{\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.}$$

□

2.4 Lebesgue Integrability and Improper Riemann Integrals

Dominated Convergence Principle

Assume f is Riemann integrable on every finite interval $[a, b]$, and assume

$$\boxed{\int_{[a,\infty)} |f| \, dm < \infty.}$$

Then the improper Riemann integral converges, and

$$\boxed{\int_a^\infty f(x) \, dx = \int_{[a,\infty)} f \, dm.}$$

Proof. For each $n \in \mathbb{N}$ with $n \geq a$, define

$$f_n = f \mathbf{1}_{[a,n]}.$$

Then

$$f_n(x) \to f(x) \qquad \text{pointwise on } [a, \infty).$$

Also,

$$|f_n(x)| = |f(x)| \mathbf{1}_{[a,n]}(x) \leq |f(x)|.$$

By assumption,

$$|f| \in L^1([a, \infty)).$$

Therefore, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_{[a,\infty)} f_n \, dm = \int_{[a,\infty)} f \, dm.$$

Since $f_n = 0$ outside $[a, n]$,

$$\int_{[a,\infty)} f_n \, dm = \int_{[a,n]} f \, dm.$$

Because f is Riemann integrable on $[a, n]$,

$$\int_{[a,n]} f \, dm = \int_a^n f(x) \, dx.$$

Hence

$$\lim_{n\to\infty} \int_a^n f(x) \, dx = \int_{[a,\infty)} f \, dm.$$

The limit on the left is exactly the improper Riemann integral. Therefore,

$$\int_a^\infty f(x) \, dx = \int_{[a,\infty)} f \, dm.$$

$\square$

Absolute Convergence

Suppose f is Riemann integrable on every compact interval $[a, b]$. If

$$\int_a^\infty |f(x)|\,dx < \infty$$

as an improper Riemann integral, then

$$f \in L^1([a, \infty)),$$

and

$$\boxed{\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.}$$

Proof. Apply the nonnegative case to $|f|$. Since $|f| \geq 0$ and

$$\int_a^\infty |f(x)|\,dx < \infty,$$

we obtain

$$\int_{[a,\infty)} |f|\,dm < \infty.$$

Thus $|f| \in L^1([a, \infty))$, meaning $f \in L^1([a, \infty))$. Then the dominated convergence argument above gives

$$\int_a^\infty f(x)\,dx = \int_{[a,\infty)} f\,dm.$$

□

3 A Caution: Conditional Improper Convergence

Important Warning

An improper Riemann integral may converge even when the function is not Lebesgue integrable.
A classical example is

$$f(x) = \frac{\sin x}{x}, \qquad x > 0.$$

The improper Riemann integral is defined by

$$\int_0^\infty \frac{\sin x}{x}\,dx := \lim_{T\to\infty} \int_0^T \frac{\sin x}{x}\,dx,$$

provided this limit exists.

3.1 Why $\int_0^\infty \frac{\sin x}{x}\,dx$ Converges

Split the integral into two parts:

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx.$$

Step 1: Convergence Near 0

We know that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Therefore $\sin x/x$ is bounded near 0. Since bounded functions are integrable on finite intervals, the integral

$$\int_0^1 \frac{\sin x}{x}\,dx$$

converges.

Step 2: Convergence on $[1,\infty)$

For $T > 1$, consider

$$\int_1^T \frac{\sin x}{x}\,dx.$$

Use integration by parts. Choose

$$u = \frac{1}{x}, \qquad dv = \sin x\,dx.$$

Then

$$du = -\frac{1}{x^2}\,dx, \qquad v = -\cos x.$$

By the integration-by-parts formula,

$$\int u\,dv = uv - \int v\,du.$$

Hence

$$\int_1^T \frac{\sin x}{x}\,dx = \left[-\frac{\cos x}{x}\right]_1^T - \int_1^T \frac{\cos x}{x^2}\,dx.$$

Now evaluate the boundary term:

$$\left[-\frac{\cos x}{x}\right]_1^T = -\frac{\cos T}{T} - \left(-\frac{\cos 1}{1}\right) = -\frac{\cos T}{T} + \cos 1.$$

Therefore,

$$\int_1^T \frac{\sin x}{x}\,dx = -\frac{\cos T}{T} + \cos 1 - \int_1^T \frac{\cos x}{x^2}\,dx.$$

As $T \to \infty$,

$$-\frac{\cos T}{T} \to 0,$$

because $|\cos T| \leq 1$. Also,

$$\left|\frac{\cos x}{x^2}\right| \leq \frac{1}{x^2},$$

and

$$\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^\infty = 1 < \infty.$$

So

$$\int_1^\infty \frac{\cos x}{x^2}\,dx$$

converges absolutely. Therefore

$$\int_1^\infty \frac{\sin x}{x}\,dx$$

converges.

Combining the convergence on $(0, 1]$ and on $[1, \infty)$, we get

$$\boxed{\int_0^\infty \frac{\sin x}{x}\,dx \text{ converges as an improper Riemann integral.}}$$

3.2 Why $\frac{\sin x}{x}$ is Not Lebesgue Integrable

Lebesgue integrability requires absolute integrability:

$$\int_0^\infty \left|\frac{\sin x}{x}\right|\,dx < \infty.$$

We show that this condition fails.

For $k = 1, 2, 3, \ldots$, consider the interval

$$[(k-1)\pi, k\pi].$$

On this interval, we have

$$x \leq k\pi.$$

Since $x > 0$, this implies

$$\frac{1}{x} \geq \frac{1}{k\pi}.$$

Therefore,

$$\left|\frac{\sin x}{x}\right| = \frac{|\sin x|}{x} \geq \frac{|\sin x|}{k\pi}.$$

Integrating over $[(k-1)\pi, k\pi]$, we get

$$\int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{x}\right|\,dx \geq \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin x|\,dx.$$

Now

$$\int_{(k-1)\pi}^{k\pi} |\sin x|\,dx = 2.$$

Hence

$$\int_{(k-1)\pi}^{k\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{k\pi}.$$

Adding these inequalities for $k = 1, 2, \dots, n$, we obtain

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{\pi} \sum_{k=1}^{n} \frac{1}{k}.$$

The harmonic series diverges:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Therefore,

$$\lim_{n\to\infty} \int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \infty.$$

So

$$\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx = \infty.$$

Thus $\sin x/x \notin L^1((0,\infty))$.

Final Conclusion

The function $\sin x/x$ has a convergent improper Riemann integral, but it is not Lebesgue integrable on $(0,\infty)$, because its absolute value has infinite integral.

$$\boxed{\text{Improper convergence is not the same as Lebesgue integrability.}}$$

4 Exercises: Dominated Convergence

Exercises 1–6

Use the Lebesgue Dominated Convergence Theorem to write the limit of integrals as a Lebesgue integral of an integrable function, without a limit sign.

1. Find

$$\lim_{n\to\infty} \int_1^n \frac{\sin x}{x^2}\, dx.$$

Solution. Define

$$f_n(x) = \frac{\sin x}{x^2} \mathbf{1}_{[1,n]}(x), \qquad x \in [1,\infty).$$

Then

$$\int_1^n \frac{\sin x}{x^2}\, dx = \int_{[1,\infty)} f_n\, dm.$$

For each fixed $x \geq 1$, eventually $x \leq n$, so

$$f_n(x) \to \frac{\sin x}{x^2}.$$

Also,

$$|f_n(x)| \le \frac{1}{x^2}, \qquad \int_1^\infty \frac{1}{x^2}\,dx = 1 < \infty.$$

Thus $1/x^2$ is an integrable dominating function. By the Dominated Convergence Theorem,

$$\boxed{\lim_{n\to\infty} \int_1^n \frac{\sin x}{x^2}\,dx = \int_{[1,\infty)} \frac{\sin x}{x^2}\,dm.}$$

$\square$

2. Find

$$\lim_{n\to\infty} \int_0^n e^{-x} \cos x\,dx.$$

Solution. Define

$$f_n(x) = e^{-x} \cos x\,\mathbf{1}_{[0,n]}(x), \qquad x \in [0,\infty).$$

Then

$$\int_0^n e^{-x} \cos x\,dx = \int_{[0,\infty)} f_n\,dm.$$

For every fixed $x \ge 0$, eventually $x \le n$, hence

$$f_n(x) \to e^{-x} \cos x.$$

Moreover,

$$|f_n(x)| \le e^{-x}, \qquad \int_0^\infty e^{-x}\,dx = 1 < \infty.$$

Therefore, by the Dominated Convergence Theorem,

$$\boxed{\lim_{n\to\infty} \int_0^n e^{-x} \cos x\,dx = \int_{[0,\infty)} e^{-x} \cos x\,dm.}$$

$\square$

3. Find

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\,dx.$$

Solution. Use the substitution

$$u = \frac{x}{n}, \qquad x = nu, \qquad dx = n\,du.$$

Then the integral becomes

$$\int_0^\infty n(1+u)^{-n} \sin u\,du.$$

Define

$$g_n(u) = n(1+u)^{-n} \sin u, \qquad u \in [0,\infty).$$

For each fixed $u > 0$, the exponential decay of $(1+u)^{-n}$ dominates the linear factor n, so

$$g_n(u) \to 0.$$

Also $g_n(0) = 0$. Hence $g_n \to 0$ pointwise on $[0, \infty)$.

We now dominate g_n. For $0 \le u \le 1$, use $|\sin u| \le u$ and $\log(1 + u) \ge u/2$. Then

$$|g_n(u)| \le nu(1 + u)^{-n} = nue^{-n \log(1+u)} \le nue^{-nu/2}.$$

The function $te^{-t/2}$ is bounded for $t \ge 0$, so $|g_n(u)| \le C$ on $[0, 1]$. For $u \ge 1$,

$$|g_n(u)| \le \frac{n}{(1 + u)^n} \le \frac{4}{(1 + u)^2}, \qquad n \ge 2.$$

Thus g_n is dominated by the integrable function

$$G(u) = C\mathbf{1}_{[0,1]}(u) + \frac{4}{(1 + u)^2}\mathbf{1}_{[1,\infty)}(u).$$

By the Dominated Convergence Theorem,

$$\boxed{\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx = \int_{[0,\infty)} 0\, dm = 0.}$$

$\square$

4. Find

$$\lim_{n\to\infty} \int_0^1 \frac{x}{1 + n^2x^2}\, dx.$$

Solution. Let

$$f_n(x) = \frac{x}{1 + n^2x^2}, \qquad x \in [0, 1].$$

For $x = 0$, $f_n(0) = 0$. For $x > 0$, $n^2x^2 \to \infty$, so

$$f_n(x) \to 0.$$

Also,

$$0 \le f_n(x) \le x,$$

and $x \in L^1([0, 1])$. By the Dominated Convergence Theorem,

$$\boxed{\lim_{n\to\infty} \int_0^1 \frac{x}{1 + n^2x^2}\, dx = \int_{[0,1]} 0\, dm = 0.}$$

$\square$

5. Find

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{1 + n^2x^2}\, dx.$$

Solution. Let

$$f_n(x) = \frac{nx}{1 + n^2x^2}, \qquad x \in [0, 1].$$

For $x = 0$, $f_n(0) = 0$. For $x > 0$,

$$\frac{nx}{1 + n^2x^2} = \frac{1/(nx)}{1/(n^2x^2) + 1} \to 0.$$

Thus $f_n \to 0$ pointwise on $[0,1]$. Now set $t = nx$. Since

$$0 \le \frac{t}{1+t^2} \le \frac{1}{2} \qquad (t \ge 0),$$

we have

$$0 \le f_n(x) \le \frac{1}{2}.$$

The constant function $1/2$ is integrable on $[0,1]$. Therefore, by the Dominated Convergence Theorem,

$$\boxed{\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\,dx = \int_{[0,1]} 0\,dm = 0.}$$

☐

6. Find

$$\lim_{n\to\infty} \int_1^n \frac{\cos(x/n)}{x^2}\,dx.$$

Solution. Define

$$f_n(x) = \frac{\cos(x/n)}{x^2}\mathbf{1}_{[1,n]}(x), \qquad x \in [1,\infty).$$

Then

$$\int_1^n \frac{\cos(x/n)}{x^2}\,dx = \int_{[1,\infty)} f_n\,dm.$$

For fixed $x \ge 1$, eventually $x \le n$, and $\cos(x/n) \to 1$. Hence

$$f_n(x) \to \frac{1}{x^2}.$$

Also,

$$|f_n(x)| \le \frac{1}{x^2}, \qquad \frac{1}{x^2} \in L^1([1,\infty)).$$

By the Dominated Convergence Theorem,

$$\boxed{\lim_{n\to\infty} \int_1^n \frac{\cos(x/n)}{x^2}\,dx = \int_{[1,\infty)} \frac{1}{x^2}\,dm = 1.}$$

☐

Exercises 7–10

Evaluate each limit. Use the Dominated Convergence Theorem when it applies. If it does not apply, compute the integral directly.

7. Evaluate

$$\lim_{n\to\infty} \int_0^n e^{-x}\sin x\,dx.$$

Solution. Define

$$f_n(x) = e^{-x}\sin x\,\mathbf{1}_{[0,n]}(x), \qquad x \in [0,\infty).$$

Then $f_n(x) \to e^{-x}\sin x$, and

$$|f_n(x)| \le e^{-x} \in L^1([0,\infty)).$$

By the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_0^n e^{-x}\sin x\,dx = \int_0^\infty e^{-x}\sin x\,dx.$$

Compute the improper integral by integration by parts. Let

$$I = \int_0^\infty e^{-x}\sin x\,dx.$$

An antiderivative is

$$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\sin x + \cos x).$$

Therefore,

$$I = \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^\infty = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}.$$

Hence

$$\boxed{\lim_{n\to\infty}\int_0^n e^{-x}\sin x\,dx = \frac{1}{2}.}$$

□

8. Evaluate

$$\lim_{n\to\infty}\int_0^\infty (1+nx^2)(1+x^2)^{-n}\,dx.$$

Solution. Let

$$f_n(x) = (1+nx^2)(1+x^2)^{-n}, \qquad x \ge 0.$$

For $x = 0$, $f_n(0) = 1$. For $x > 0$, the factor $(1+x^2)^{-n}$ decays exponentially while $1+nx^2$ grows only linearly, so

$$f_n(x) \to 0.$$

Thus the pointwise limit is 1 at $x = 0$ and 0 for $x > 0$, which is equal to 0 almost everywhere.

We dominate the sequence. For $0 \le x \le 1$, since $\log(1+x^2) \ge x^2/2$,

$$f_n(x) \le (1+nx^2)e^{-nx^2/2} \le C.$$

For $x \ge 1$ and $n \ge 2$,

$$f_n(x) \le \frac{1+2x^2}{(1+x^2)^2}.$$

Hence f_n is dominated by

$$G(x) = C\mathbf{1}_{[0,1]}(x) + \frac{1+2x^2}{(1+x^2)^2}\mathbf{1}_{[1,\infty)}(x),$$

which is integrable on $[0, \infty)$. By the Dominated Convergence Theorem,

$$\boxed{\lim_{n\to\infty} \int_0^\infty (1+nx^2)(1+x^2)^{-n}\, dx = 0.}$$

□

9. Evaluate

$$\lim_{n\to\infty} \int_0^1 \left(1+\frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)\, dx.$$

Solution. Let

$$f_n(x) = \left(1+\frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right), \qquad x \in [0,1].$$

For fixed $x \in [0,1]$,

$$\left(1+\frac{x}{n}\right)^{-n} \to e^{-x}, \qquad \cos\left(\frac{x}{n}\right) \to 1.$$

Thus $f_n(x) \to e^{-x}$. Also,

$$|f_n(x)| \le 1,$$

and $1 \in L^1([0,1])$. By the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_0^1 \left(1+\frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)\, dx = \int_0^1 e^{-x}\, dx.$$

Therefore,

$$\boxed{\lim_{n\to\infty} \int_0^1 \left(1+\frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)\, dx = 1 - e^{-1}.}$$

□

10. Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{n}{1+n^2x^2}\, dx.$$

Solution. For each fixed $x > 0$,

$$\frac{n}{1+n^2x^2} \to 0,$$

but at $x = 0$, the value is n. The functions form a spike near 0, so the Dominated Convergence Theorem does not apply.

Compute directly. Let

$$u = nx, \qquad du = n\, dx.$$

Then

$$\int_0^1 \frac{n}{1+n^2x^2}\, dx = \int_0^n \frac{1}{1+u^2}\, du = [\arctan u]_0^n = \arctan n.$$

Taking the limit,

$$\boxed{\lim_{n\to\infty} \int_0^1 \frac{n}{1+n^2x^2}\, dx = \frac{\pi}{2}.}$$

Warning

Although the functions converge to 0 almost everywhere, their integrals do not converge to 0. This shows why the dominating function in the Dominated Convergence Theorem is essential.

□

5 Exercises: Riemann Integrability by the Lebesgue Criterion

Exercises

Use the Lebesgue criterion for Riemann integrability: a bounded function on $[a, b]$ is Riemann integrable if and only if its set of discontinuities has measure zero. Exercise 34 was mentioned in the heading but was not included in the prompt, so the solutions below cover Exercises 28–33.

1. Determine whether

$$\int_0^{2\pi} 4\sin\left(\frac{1}{x}\right) dx$$

exists.

Solution. The expression $4\sin(1/x)$ is not defined at $x = 0$. For the ordinary Riemann integral on $[0, 2\pi]$, we may assign any finite value to the function at 0; changing one point does not affect Riemann integrability.

On $(0, 2\pi]$, the function

$$f(x) = 4\sin\left(\frac{1}{x}\right)$$

is continuous because it is a composition of continuous functions. At $x = 0$, the function oscillates and has no limit, since $\sin(1/x)$ oscillates between -1 and 1 infinitely often near 0. Therefore the only point of discontinuity is

$$D_f = \{0\}.$$

Also,

$$|f(x)| \leq 4,$$

so f is bounded. Since $\{0\}$ has measure zero, the Lebesgue criterion implies that f is Riemann integrable.

Conclusion

$$\boxed{\int_0^{2\pi} 4\sin\left(\frac{1}{x}\right) dx \text{ exists after defining } f(0) \text{ arbitrarily.}}$$

□

2. Determine whether

$$\int_0^1 f(x)\,dx,$$

where

$$f(x) = \begin{cases} 1, & x \in [0, 0.5), \\ 0, & x \in [0.5, 1). \end{cases}$$

exists.

Solution. The function is constant on $[0, 0.5)$ and constant on $[0.5, 1)$. Therefore it is continuous at every point except possibly the jump point $x = 0.5$, and possibly at $x = 1$ if no value is specified there.

At $x = 0.5$, the left-hand limit is 1, while the right-hand limit and the value are 0. Hence f is discontinuous at 0.5.

If we define $f(1) = 0$, then f is continuous at the right endpoint 1. Even if another value is assigned at 1, that only adds one more discontinuity. Thus the set of discontinuities is finite, for example

$$D_f = \{0.5\}$$

if $f(1) = 0$. A finite set has measure zero, and the function is bounded. Hence f is Riemann integrable on $[0, 1]$.

$$\int_0^1 f(x) dx \text{exists.}$$

□

3. Determine whether

$$\int_0^4 f(x)\,dx,$$

where

$$f(x) = \begin{cases} 1, & x \in [0, 1), \\ x, & x \in [1, 2), \\ x^2, & x \in [2, 3), \\ \ln x, & x \in [3, 4] \end{cases}$$

exists.

Solution. Each formula is continuous on the interval where it is used. Therefore possible discontinuities can occur only at the joining points

$$x = 1, \qquad x = 2, \qquad x = 3.$$

At $x = 1$:

$$\lim_{x \to 1^-} f(x) = 1, \qquad f(1) = 1, \qquad \lim_{x \to 1^+} f(x) = 1.$$

So f is continuous at 1.

At $x = 2$:

$$\lim_{x\to 2^-} f(x) = 2, \qquad f(2) = 2^2 = 4, \qquad \lim_{x\to 2^+} f(x) = 4.$$

The left-hand limit does not equal the value, so f is discontinuous at 2.

At $x = 3$:

$$\lim_{x\to 3^-} f(x) = 3^2 = 9, \qquad f(3) = \ln 3, \qquad \lim_{x\to 3^+} f(x) = \ln 3.$$

The left-hand limit does not equal the value, so f is discontinuous at 3.

Thus

$$D_f = \{2, 3\}.$$

The function is bounded on $[0, 4]$, and D_f is finite, hence has measure zero. By the Lebesgue criterion, f is Riemann integrable.

Conclusion

$$\boxed{\int_0^4 f(x)\,dx \text{ exists.}}$$

□

4. *Solution.*

$$\int_0^1 f(x)\,dx,$$

where

$$f(x) = \begin{cases} 1, & x = \dfrac{1}{2^n} \text{ for integers } n, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$A = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\} \cap [0, 1].$$

We have

$$A = \left\{ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \right\}.$$

Thus

$$f(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

The set A is countable, and its only accumulation point in $[0, 1]$ is 0.

We now identify the points of discontinuity.

If $x \in A$, then

$$f(x) = 1.$$

However, every interval around x contains points not in A, where $f = 0$. Therefore f is discontinuous at every point of A.

At $x = 0$, we have

$$f(0) = 0.$$

But

$$\frac{1}{2^n} \to 0$$

and

$$f\left(\frac{1}{2^n}\right) = 1.$$

Hence f is discontinuous at 0.

Now suppose

$$x \notin A \cup \{0\}.$$

Then x is not an accumulation point of A. Hence there exists a small interval around x containing no points of A. On that interval,

$$f = 0.$$

Therefore f is continuous at x.

Thus the set of discontinuities is

$$D_f = A \cup \{0\} = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right\} \cup \{0\}.$$

This set is countable, so it has measure zero. Also,

$$0 \le f(x) \le 1,$$

so f is bounded.

By the Riemann–Lebesgue criterion, f is Riemann integrable on $[0, 1]$.

Since $f = 1$ only on a countable set and $f = 0$ elsewhere,

$$\int_0^1 f(x)\,dx = 0.$$

Therefore,

$$\boxed{\int_0^1 f(x)\,dx \text{ exists and equals } 0.}$$

□