Simple Introduction to the Lebesgue Integral

1 Lecture 1: Outer Measure

In calculus, we know how to find the length of an interval. Measure theory begins with this idea and extends it to more general sets.

Length

Let $I = (a, b)$ be an open interval. Its length is defined by

$$\ell(I) = b - a.$$

We also define

$$\ell(\emptyset) = 0.$$

Let $A \subset \mathbb{R}$. Then:

(i) If A is a **finite disjoint union** of intervals,

$$A = \bigcup_{k=1}^{n} I_k, \quad I_i \cap I_j = \emptyset \ (i \neq j),$$

then

$$\ell(A) = \sum_{k=1}^{n} \ell(I_k).$$

(ii) More generally, if A is a **countable disjoint union** of intervals,

$$A = \bigcup_{n=1}^{\infty} I_n,$$

then

$$\ell(A) = \sum_{n=1}^{\infty} \ell(I_n).$$

(length = sum of lengths of disjoint pieces)

Definition of Outer Measure

Some sets are not intervals, so we cover them by open intervals.

Definition of Outer Measure

For any set $A \subset \mathbb{R}$, the *outer measure* of A is defined by

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) : A \subset \bigcup_{k=1}^{\infty} I_k,\ I_k \text{ open intervals} \right\}.$$

(smallest total length of all coverings of A)

This means:

- cover A by open intervals,
- add their lengths,
- take the smallest possible total.

Every finite set has outer measure zero.

Proof.

Let

$$S = \{a_1, a_2, \dots, a_n\}.$$

Take $\varepsilon > 0$. Around each point a_k, choose the interval

$$I_k = \left(a_k - \frac{\varepsilon}{2n},\ a_k + \frac{\varepsilon}{2n}\right).$$

Then

$$S \subset \bigcup_{k=1}^{n} I_k,$$

and each interval has length

$$\ell(I_k) = \frac{\varepsilon}{n}.$$

So

$$\sum_{k=1}^{n} \ell(I_k) = n \cdot \frac{\varepsilon}{n} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude

$$m^*(S) = 0.$$

Every countable set has outer measure zero.

Proof.
Let

$$S = \{a_1, a_2, a_3, \dots\}.$$

Take $\varepsilon > 0$. Around each point a_n, choose the interval

$$I_n = \left(a_n - \frac{\varepsilon}{2^{n+1}},\, a_n + \frac{\varepsilon}{2^{n+1}} \right).$$

Then

$$S \subset \bigcup_{n=1}^{\infty} I_n,$$

and

$$\ell(I_n) = \frac{\varepsilon}{2^n}.$$

So

$$\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude

$$m^*(S) = 0.$$

Example.
The sets

$$\mathbb{N}, \qquad \mathbb{Z}, \qquad \mathbb{Q}$$

all have outer measure zero.

The interval $[0,1]$

The interval $[0,1]$ has outer measure 1.

We claim that

$$m^*([0,1]) = 1.$$

Proof.
First, let $\varepsilon > 0$. Then

$$[0,1] \subset (-\varepsilon, 1+\varepsilon),$$

so

$$m^*([0,1]) \leq 1 + 2\varepsilon.$$

Since ε can be made arbitrarily small,

$$m^*([0,1]) \leq 1.$$

Now take any open cover of $[0,1]$. Because $[0,1]$ is compact, there is a finite subcover. That finite cover must have total length at least 1. Hence

$$m^*([0,1]) \geq 1.$$

Combining both inequalities, we conclude

$$m^*([0,1]) = 1.$$

Basic Properties

Basic Properties of Outer Measure.

For all sets $A, B, E_1, E_2, \dots \subset \mathbb{R}$:

1. $m^*(A) \geq 0$,
2. If $A \subset B$, then $m^*(A) \leq m^*(B)$,
3. $$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Proof.

1. By definition, $m^*(A)$ is the infimum of sums of lengths of intervals, and each length is nonnegative. Hence
$$m^*(A) \geq 0.$$
2. Suppose $A \subset B$. Any cover of B by open intervals also covers A. Therefore, the collection of coverings of A is larger, so
$$m^*(A) \leq m^*(B).$$
3. Let $\varepsilon > 0$. For each k, choose a countable collection of open intervals $\{I_{k,j}\}_{j=1}^{\infty}$ such that
$$E_k \subset \bigcup_{j=1}^{\infty} I_{k,j} \quad \text{and} \quad \sum_{j=1}^{\infty} \ell(I_{k,j}) \leq m^*(E_k) + \frac{\varepsilon}{2^k}.$$
Then
$$\bigcup_{k=1}^{\infty} E_k \subset \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{\infty} I_{k,j}.$$
So this gives a cover of $\bigcup_{k=1}^{\infty} E_k$, and therefore
$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \ell(I_{k,j}).$$
Hence
$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \left(m^*(E_k) + \frac{\varepsilon}{2^k}\right) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, we conclude
$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Intervals Again

Outer measure of intervals.

$$m^*([a,b]) = m^*((a,b)) = m^*([a,b)) = m^*((a,b]) = b - a.$$

The endpoints do not affect the outer measure, since finite sets (such as $\{a\}$ or $\{b\}$) have outer measure zero.

Equal Almost Everywhere

Equal Almost Everywhere.

In Lebesgue theory, changing a function on a set of measure zero does not affect its value in most contexts (especially integration).

Definition. Let f and g be functions on the same domain D. We say that f and g are *equal almost everywhere* if

$$f(x) = g(x)$$

for all $x \in D$ except on a set of measure zero. We write

$$f = g \quad \text{a.e.}$$

How to check. To show that $f = g$ a.e.:

1. define the set
$$S = \{x \in D : f(x) \neq g(x)\},$$
2. show that S has measure zero.

Example 1 (Dirichlet-type).
Define on $[0, 1]$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases} \qquad g(x) = 0.$$

Then f and g differ only on $\mathbb{Q} \cap [0, 1]$, which is countable and has measure zero. Hence

$$f = g \quad \text{a.e.}$$

Example 2.
Let

$$f(x) = 1, \qquad g(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases} \quad x \in [0, 1].$$

Then

$$S = [0, 1] \setminus \mathbb{Q},$$

which does *not* have measure zero. Hence

$$f \neq g \quad \text{a.e.}$$

Example 3.
Let

$$f(x) = x, \qquad g(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ x, & x \notin \mathbb{Q}, \end{cases} \quad x \in [0, 1].$$

Then

$$S = \mathbb{Q} \cap [0, 1],$$

which is countable, so

$$f = g \quad \text{a.e.}$$

Remark. Later we will see that if $f = g$ a.e., then

$$\int_0^1 f\, dm = \int_0^1 g\, dm.$$

2 Exercises

We use the following facts:

- The outer measure of an interval is its length.
- Endpoints do not affect outer measure.
- Finite and countable sets have outer measure zero.
- Disjoint intervals add.

1–8 Find the Outer Measure

1. $[0, 2]$
$$m^*([0, 2]) = 2.$$
2. $(1, 3)$
$$m^*((1, 3)) = 2.$$
3. $(-3, 21.4]$
$$m^*((-3, 21.4]) = 24.4.$$
4. $[-2\pi, \pi + 6)$
$$m^* = 3\pi + 6.$$
5. $\left(-\sqrt{5}, \sqrt{3}\right]$
$$m^* = \sqrt{3} + \sqrt{5}.$$
6. $(-3, 2] \cup [6, 9]$
$$m^* = 5 + 3 = 8.$$
7. $\{-6, -\sqrt{2}, 3, 7, 31, e, \pi\}$
$$m^* = 0.$$
8. $\{2, 4, 6, 8, 10, \dots\}$
$$m^* = 0.$$

9–15 Show They Have Outer Measure Zero

9. $\{-3, 7, 31\}$

   Finite set $\Rightarrow m^* = 0$.
10. $\{-6, -\sqrt{2}, 3, 7, 31, e, \pi\}$

    Finite set $\Rightarrow m^* = 0$.
11. $\{-20, -19, \dots, 5\}$

    Finite set $\Rightarrow m^* = 0$.
12. $\mathbb{Z}$

    Countable $\Rightarrow m^* = 0$.

13. $\{2, 4, 6, 8, 10, \dots\}$
    Countable $\Rightarrow m^* = 0$.

14. $\{\dots, -4, -2, 0, 2, 4, \dots\}$
    Countable $\Rightarrow m^* = 0$.

15. $\mathbb{Q} \cap (0, 2]$
    Countable $\Rightarrow m^* = 0$.

16–19 Equality Almost Everywhere

16. The functions differ only at the endpoints:

$$S = \{0, 1\}.$$

Since S is finite,

$$m^*(S) = 0.$$

Hence

$$f = g \quad \text{a.e.}$$

17. The only extra point is $x = 0$:

$$S = \{0\}.$$

Since a single point has outer measure zero,

$$f = g \quad \text{a.e.}$$

18. For $x \neq 0$,

$$\frac{x^2 - x}{x} = x - 1.$$

So

$$S = \{0\}.$$

Hence

$$f = g \quad \text{a.e.}$$

19. For $x \neq -2$,

$$\frac{x^2 - 4}{x + 2} = x - 2.$$

So

$$S = \{-2\}.$$

Hence

$$f = g \quad \text{a.e.}$$

3 Lecture 2: σ-Algebra, Borel Sets, and Measure

σ-Algebra

In Lecture 1, we introduced outer measure to assign sizes to sets. This leads to a natural question:

> Which sets can we measure?

To answer this, we introduce the concept of a *σ-algebra*.

Definition (σ-algebra).
Let X be a set. A collection $\mathcal{S}$ of subsets of X is called a **σ-algebra** if:

- $\emptyset \in \mathcal{S}$,
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$,
- if $E_1, E_2, E_3, \dots \in \mathcal{S}$, then
$$\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}.$$

The pair $(X, \mathcal{S})$ is called a **measurable space**.

Examples.

- $\{\emptyset, X\}$ (smallest σ-algebra),
- $\mathcal{P}(X)$ (all subsets of X),
- $\{E \subseteq X : E \text{ is countable or } X \setminus E \text{ is countable}\}$.

Basic Properties of a σ-Algebra

Let $\mathcal{S}$ be a σ-algebra on a set X. Then:

1. $X \in \mathcal{S}$,
2. If $D, E \in \mathcal{S}$, then
$$D \cup E \in \mathcal{S}, \qquad D \cap E \in \mathcal{S}, \qquad D \setminus E \in \mathcal{S}.$$
3. If $E_1, E_2, E_3, \dots \in \mathcal{S}$, then
$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}.$$

Proof.

(1) Since $\emptyset \in \mathcal{S}$ and
$$X = X \setminus \emptyset,$$
closure under complements implies $X \in \mathcal{S}$.

(2) Let $D, E \in \mathcal{S}$.
Since $\mathcal{S}$ is closed under countable unions,
$$D \cup E \in \mathcal{S}.$$
Using De Morgan's law,
$$X \setminus (D \cap E) = (X \setminus D) \cup (X \setminus E).$$
The right-hand side belongs to $\mathcal{S}$, so taking complements gives
$$D \cap E \in \mathcal{S}.$$
Finally,
$$D \setminus E = D \cap (X \setminus E),$$
so
$$D \setminus E \in \mathcal{S}.$$

(3) Let $E_1, E_2, E_3, \dots \in \mathcal{S}$. By De Morgan's law,
$$X \setminus \left(\bigcap_{k=1}^{\infty} E_k \right) = \bigcup_{k=1}^{\infty} (X \setminus E_k).$$
Each $X \setminus E_k$ belongs to $\mathcal{S}$, and $\mathcal{S}$ is closed under countable unions. Hence the right-hand side belongs to $\mathcal{S}$. Taking complements gives
$$\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}.$$

Borel σ-Algebra on $\mathbb{R}$

The most important σ-algebra on the real line is the *Borel σ-algebra*.

Definition.
The **Borel σ-algebra** on $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$, is the smallest σ-algebra that contains all open intervals (a, b).
A set in $\mathcal{B}(\mathbb{R})$ is called a **Borel set**.

Examples of Borel Sets.

1. **Open intervals:**
$$(a, b) \in \mathcal{B}(\mathbb{R}).$$
2. **Closed intervals:**
$$[a, b] = \mathbb{R} \setminus \big((-\infty, a) \cup (b, \infty)\big).$$
3. **Half-open intervals:**
$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right).$$
4. **Single points:**
$$\{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a + \frac{1}{n}\right).$$
5. **Countable sets:** if
$$E = \{x_1, x_2, x_3, \dots\}, \quad E = \bigcup_{n=1}^{\infty} \{x_n\}.$$
6. **unbounded intervals:**
$$(a, \infty),\ [a, \infty) \in \mathcal{B}(\mathbb{R}).$$

Measure

After choosing a σ-algebra, we can define a notion of size for sets.

Definition (Measure).
Let $\mathcal{S}$ be a σ-algebra on a set X. A function

$$\mu : \mathcal{S} \to [0, \infty]$$

is called a **measure** if:

1. $$\mu(\emptyset) = 0,$$

2. For every sequence of pairwise disjoint sets $E_1, E_2, E_3, \ldots$ in $\mathcal{S}$,

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

The triple $(X, \mathcal{S}, \mu)$ is called a **measure space**.

Examples of Measures.
(1) Counting Measure.

$$\mu(E) = \begin{cases} \text{number of elements of } E, & \text{if } E \text{ is finite,} \\ \infty, & \text{if } E \text{ is infinite.} \end{cases}$$

This measure counts the number of elements in a set. For example,

$$\mu(\{1, 2, 4\}) = 3, \qquad \mu(\mathbb{N}) = \infty.$$

(2) Dirac Measure.
Fix a point $c \in \mathbb{R}$. Define

$$\delta_c(E) = \begin{cases} 1, & \text{if } c \in E, \\ 0, & \text{if } c \notin E. \end{cases}$$

This measure concentrates all mass at a single point. For example,

$$\delta_5([4, 6]) = 1, \qquad \delta_5((0, 4)) = 0.$$

A Set Function That Is Not a Measure

Define

$$\mu(E) = \begin{cases} 0, & \text{if } E \text{ is finite,} \\ \infty, & \text{if } E \text{ is infinite.} \end{cases}$$

This is *not* a measure.
Indeed, let

$$E_n = \{n\}, \qquad n = 1, 2, 3, \ldots$$

Then

$$\mu(E_n) = 0 \qquad \text{for all } n,$$

so

$$\sum_{n=1}^{\infty} \mu(E_n) = 0.$$

But

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{N},$$

and ℕ is infinite, so

$$\mu(\mathbb{N}) = \infty.$$

Therefore

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \neq \sum_{n=1}^{\infty} \mu(E_n).$$

So countable additivity fails.

Lecture 3: Lebesgue Measure

Lebesgue Measurable Sets

In Lecture 1, we defined the *outer measure* m^*. In Lecture 2, we introduced σ-algebras and measures.

Now we ask:

> Which sets are good enough so that outer measure behaves like a true measure?

These sets are called **Lebesgue measurable sets**.

Definition (Lebesgue measurable set).
A set $E \subset \mathbb{R}$ is called **Lebesgue measurable** if, for every set $A \subset \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c),$$

where $E^c = \mathbb{R} \setminus E$.
This condition is known as the **Carathéodory criterion**.

Interpretation.
The set E splits any set A into two parts:

$$A \cap E \qquad \text{and} \qquad A \cap E^c.$$

If E is measurable, then the outer measure of A is exactly the sum of the outer measures of these two parts.

Key Observation.
Outer measure always satisfies

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c),$$

since

$$A = (A \cap E) \cup (A \cap E^c).$$

Therefore, to prove that E is measurable, it is enough to show

$$m^*(A \cap E) + m^*(A \cap E^c) \leq m^*(A).$$

First Examples of Measurable Sets

First Examples of Lebesgue Measurable Sets.

Theorem 1. The sets $\emptyset$ and $\mathbb{R}$ are measurable.

Proof.
Let $A \subset \mathbb{R}$.
For $E = \emptyset$,

$$A \cap \emptyset = \emptyset, \qquad A \cap \emptyset^c = A.$$

Hence

$$m^*(A \cap \emptyset) + m^*(A \cap \emptyset^c) = 0 + m^*(A) = m^*(A).$$

For $E = \mathbb{R}$,

$$A \cap \mathbb{R} = A, \qquad A \cap \mathbb{R}^c = \emptyset.$$

Thus

$$m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c) = m^*(A) + 0 = m^*(A).$$

Theorem 2. A set E is measurable if and only if its complement E^c is measurable.

Proof.
If E is measurable, then for every $A \subset \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

This expression is symmetric in E and E^c, so E^c is also measurable.

Theorem 3. Every set of outer measure zero is measurable.

Proof.
Let $E \subset \mathbb{R}$ with $m^*(E) = 0$, and let $A \subset \mathbb{R}$.
Since $A \cap E \subset E$, by monotonicity,

$$m^*(A \cap E) \leq m^*(E) = 0,$$

so

$$m^*(A \cap E) = 0.$$

Therefore,

$$m^*(A \cap E) + m^*(A \cap E^c) = 0 + m^*(A \cap E^c) \leq m^*(A).$$

Since the reverse inequality always holds, we conclude

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Thus E is measurable.

The interval (a, ∞) is measurable.

Proof.
Let $A \subset \mathbb{R}$ and define

$$A_1 = A \cap (a, \infty), \qquad A_2 = A \cap (-\infty, a].$$

Then $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$.
We already know

$$m^*(A) \leq m^*(A_1) + m^*(A_2),$$

so it remains to prove the reverse inequality.
Assume $m^*(A) < \infty$ and let $\varepsilon > 0$. Choose intervals I_n covering A such that

$$\sum \ell(I_n) \leq m^*(A) + \varepsilon.$$

Split each I_n:

$$J_n = I_n \cap (a, \infty), \qquad K_n = I_n \cap (-\infty, a].$$

Then

$$A_1 \subset \bigcup J_n, \qquad A_2 \subset \bigcup K_n,$$

and

$$\ell(I_n) = \ell(J_n) + \ell(K_n).$$

Hence

$$m^*(A_1) + m^*(A_2) \leq m^*(A) + \varepsilon.$$

Letting $\varepsilon \to 0$ gives the result.

Closure Properties

If E and F are measurable, then $E \cup F$ is measurable.

Proof.
Let $A \subset \mathbb{R}$ and split

$$A_1 = A \cap E, \quad A_2 = A \cap E^c \cap F, \quad A_3 = A \cap E^c \cap F^c.$$

These are disjoint and

$$A = A_1 \cup A_2 \cup A_3.$$

Also,

$$A \cap (E \cup F) = A_1 \cup A_2, \qquad A \cap (E \cup F)^c = A_3.$$

Using measurability of E and F, we obtain

$$m^*(A) = m^*(A \cap (E \cup F)) + m^*(A \cap (E \cup F)^c).$$

The Collection of Measurable Sets

The collection $\mathcal{M}$ of measurable sets is a σ-algebra.

Proof.
We already know:

- $\emptyset \in \mathcal{M}$,
- $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$,
- $E, F \in \mathcal{M} \Rightarrow E \cup F \in \mathcal{M}$.

Let $E_n \in \mathcal{M}$ and define

$$F_n = \bigcup_{k=1}^{n} E_k, \qquad F = \bigcup_{k=1}^{\infty} E_k.$$

Each F_n is measurable. Since $F_n \uparrow F$, measurability passes to the limit, so $F \in \mathcal{M}$.

Borel Sets Are Measurable

Every Borel set is Lebesgue measurable.

Proof.
We proved (a, ∞) is measurable, hence $(-\infty, a]$ is measurable. Thus

$$(a, b) = (a, \infty) \cap (-\infty, b)$$

is measurable.
Every open set is a countable union of open intervals, so it is measurable.
Since $\mathcal{M}$ is a σ-algebra containing all open sets,

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{M}.$$

Lebesgue Measure

Definition (Lebesgue Measure).
For $E \in \mathcal{M}$, define

$$m(E) = m^*(E).$$

The triple $(\mathbb{R}, \mathcal{M}, m)$ is called the **Lebesgue measure space**.

Theorem. m is a measure.

Proof.

$$m(\emptyset) = 0.$$

If E_n are disjoint measurable sets, then

$$m\Big(\bigcup E_n\Big) \leq \sum m(E_n)$$

by outer measure, and measurability gives the reverse inequality.
Hence equality holds.

Basic Properties of Lebesgue Measure

Basic Properties of Lebesgue Measure.

Lebesgue measure m satisfies the following:

(a) Monotonicity: If $E \subset F$, then

$$m(E) \leq m(F).$$

(b) Countable subadditivity:

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n).$$

(c) Continuity from below: If

$$E_1 \subset E_2 \subset \cdots,$$

then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

(d) Continuity from above: If

$$E_1 \supset E_2 \supset \cdots$$

and $m(E_k) < \infty$ for some k, then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

Lecture 4: Lebesgue Measurable Functions

In the previous lecture, we studied measurable sets. Now we study measurable *functions*.

Definition (Measurable Function).

Let $E \subseteq \mathbb{R}$ be a measurable set, and let

$$f : E \to \mathbb{R}.$$

We say that f is **Lebesgue measurable** if for every real number α,

$$\{x \in E : f(x) > \alpha\}$$

is a measurable set.

Examples

Example 1 (Constant Function).

If $f(x) = c$ for all x, then f is measurable.

Proof.

For any α,

$$\{x : f(x) > \alpha\} = \begin{cases} \emptyset, & \alpha \geq c, \\ E, & \alpha < c. \end{cases}$$

Both $\emptyset$ and E are measurable.

Example 2 (Continuous Function).

Every continuous function is measurable.

Proof.

If f is continuous, then for every α,

$$\{x : f(x) > \alpha\} = f^{-1}((\alpha, \infty))$$

is open. Every open set is measurable, so f is measurable.

Example 3 (Characteristic Function).

Let $A \subseteq E$ be measurable. Define

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Then χ_A is measurable.

Proof.
For any α,

$$\{x : \chi_A(x) > \alpha\} = \begin{cases} \emptyset, & \alpha \geq 1, \\ A, & 0 \leq \alpha < 1, \\ E, & \alpha < 0. \end{cases}$$

Each of these sets is measurable.

Example 4 (Monotone Function).

Every monotone function is measurable.

Remark.
If f is monotone increasing, then for each α the set

$$\{x : f(x) > \alpha\}$$

is an interval, or all of $\mathbb{R}$, or empty. Hence it is measurable.

Basic Operations

Basic Operations on Measurable Functions.

If $f, g : E \to \mathbb{R}$ are measurable and $c \in \mathbb{R}$, then

$$cf, \qquad f+g, \qquad |f|, \qquad f^2, \qquad fg$$

are also measurable.

Remark.
Measurable functions are closed under the usual algebraic operations.

Idea of the Proof.

- For cf, rewrite $\{x : cf(x) > \alpha\}$ in terms of f.
- For $|f|$,
$$\{|f| > \alpha\} = \{f > \alpha\} \cup \{f < -\alpha\}.$$
- For $f+g$, express $\{f+g > \alpha\}$ using countable unions of measurable sets.
- For f^2,
$$\{f^2 > \alpha\} = \{f > \sqrt{\alpha}\} \cup \{f < -\sqrt{\alpha}\} \qquad (\alpha > 0).$$
- For fg, use
$$fg = \frac{1}{4}\big((f+g)^2 - (f-g)^2\big).$$

Positive and Negative Parts

For any function f, define

$$f^+(x) = \max\{f(x), 0\}, \qquad f^-(x) = \max\{-f(x), 0\}.$$

Then
$$f = f^+ - f^-, \qquad |f| = f^+ + f^-.$$

Theorem.
A function f is measurable if and only if both f^+ and f^- are measurable.

Remark.
Every function can be decomposed into its positive and negative parts.

Extended Real-Valued Functions

Sometimes a function may take the values $+\infty$ or $-\infty$. So we use the extended real line

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}.$$

Definition.
A function

$$f : E \to \overline{\mathbb{R}}$$

is measurable if for every $\alpha \in \mathbb{R}$,

$$\{x \in E : f(x) > \alpha\}$$

is measurable.

Remark.
This is the same definition as before, but now infinite values are allowed.

Almost Everywhere

Definition (Almost Everywhere).

A statement $P(x)$ is true **almost everywhere** on E if the set where it fails has measure zero.

For example,

$$f = g \quad \text{a.e. on } E$$

means

$$m(\{x \in E : f(x) \neq g(x)\}) = 0.$$

If $f = g$ almost everywhere on E, and f is measurable, then g is measurable.

Remark.
Changing a measurable function on a set of measure zero does not destroy measurability.

Supremum, Infimum, and Limits

Supremum, Infimum, and Limits.

If $f_1, f_2, f_3, \dots$ are measurable, then

$$\sup_n f_n, \qquad \inf_n f_n, \qquad \limsup_{n\to\infty} f_n, \qquad \liminf_{n\to\infty} f_n$$

are measurable.

Corollary.
If $f_n \to f$ pointwise and each f_n is measurable, then f is measurable.

Remark.
Measurable functions are closed under pointwise limits.

Simple Functions

Definition (Simple Function).

A function φ is called a **simple function** if it takes only finitely many values and can be written in the form

$$\varphi(x) = \sum_{i=1}^{N} a_i \, \chi_{A_i}(x),$$

where each A_i is measurable.

Example.
The function

$$\varphi(x) = 2\chi_{[0,2)}(x) + 4\chi_{[2,4)}(x) + 1\chi_{[4,5)}(x)$$

is a simple function.

Remark.
A simple function is a step function with finitely many steps.

Simple functions are important because every nonnegative measurable function can be approximated from below by simple functions.

Theorem.
If $f : E \to [0, \infty]$ is measurable, then there exists a sequence of nonnegative simple functions (φ_n) such that

$$\varphi_1(x) \leq \varphi_2(x) \leq \varphi_3(x) \leq \cdots$$

for every $x \in E$, and

$$\varphi_n(x) \to f(x) \qquad \text{for all } x \in E.$$

Remark.
We write

$$\varphi_n \uparrow f.$$

This means that the simple functions increase to f.

Remark.
The Lebesgue integral is first defined for simple functions and then extended to general measurable functions by approximation.

Lecture 5: Lebesgue Integral of Nonnegative Functions

In this lecture, we define the Lebesgue integral.

We begin with nonnegative functions:

$$f(x) \geq 0.$$

Sometimes a function may take the value $+\infty$, so we use the rules

$$a + \infty = \infty, \qquad a \cdot \infty = \infty \quad (a > 0), \qquad 0 \cdot \infty = 0.$$

1. Integral of a Simple Function

A **simple function** takes only finitely many values. Suppose

$$f = \sum_{i=1}^{n} a_i \, \chi_{A_i},$$

where each $a_i \geq 0$ and each A_i is measurable.
We define

$$\int_E f \, dm = \sum_{i=1}^{n} a_i \, m(A_i).$$

Remark.
This is natural:

$$\text{integral} = \text{value} \times \text{size of the set},$$

summed over all pieces.

Example.
Let

$$f(x) = \begin{cases} 2, & -1 < x < 1, \\ 3, & 3 < x < 7, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_{\mathbb{R}} f \, dm = 2 \cdot m((-1, 1)) + 3 \cdot m((3, 7)).$$

Since

$$m((-1, 1)) = 2, \qquad m((3, 7)) = 4,$$

we get

$$\int_{\mathbb{R}} f \, dm = 2 \cdot 2 + 3 \cdot 4 = 16.$$

2. Integral of a General Nonnegative Function

Definition.
If $f : E \to [0, \infty]$ is measurable, define

$$\int_E f\,dm = \sup\left\{\int_E \varphi\,dm : \varphi \text{ is simple, } 0 \le \varphi \le f\right\}.$$

Remark.
The idea is:

- choose simple functions below f,
- compute their integrals,
- take the largest possible value.

3. Basic Properties

Let $f, g : E \to [0, \infty]$ be measurable and let $\lambda \ge 0$. Then:

- If $f \le g$, then
$$\int_E f\,dm \le \int_E g\,dm.$$
- $$\int_E \lambda f\,dm = \lambda \int_E f\,dm.$$
- If $F \subset E$, then
$$\int_F f\,dm = \int_E f\chi_F\,dm.$$
- If $m(E) = 0$, then
$$\int_E f\,dm = 0.$$

Remark.
These are the basic rules expected of an integral.

4. Monotone Convergence Theorem

Monotone Convergence Theorem.

Let

$$0 \leq f_1 \leq f_2 \leq f_3 \leq \cdots$$

be measurable functions on E, and suppose

$$f_n(x) \to f(x) \quad \text{for every } x \in E.$$

Then

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Remark.
If the functions increase pointwise, then the integrals increase to the integral of the limit.

Idea of the Proof.
Since $f_1 \leq f_2 \leq \cdots \leq f$, we have

$$\int_E f_1 \, dm \leq \int_E f_2 \, dm \leq \cdots \leq \int_E f \, dm.$$

So the limit exists. Then compare simple functions below f with the functions f_n to obtain the reverse inequality.

5. Fatou's Lemma

Fatou's Lemma.

If $f_n : E \to [0, \infty]$ are measurable, then

$$\int_E \liminf_{n\to\infty} f_n \, dm \leq \liminf_{n\to\infty} \int_E f_n \, dm.$$

Remark.
The integral of the $\liminf$ is less than or equal to the $\liminf$ of the integrals.

Remark.
Fatou's lemma is often used to prove stronger convergence theorems.

Lebesgue Integrable Functions

So far we integrated only nonnegative functions. Now we allow functions that may be positive or negative.

6. Positive and Negative Parts

For a real-valued function f, define

$$f^+(x) = \max\{f(x), 0\}, \qquad f^-(x) = \max\{-f(x), 0\}.$$

Then

$$f = f^+ - f^-, \qquad |f| = f^+ + f^-.$$

Remark.
f^+ is the positive part of f, and f^- is the negative part.

7. Integrable Functions

Definition (Lebesgue Integrable Function).

Let $f : E \to \mathbb{R}$ be measurable.
We say that f is **Lebesgue integrable** if

$$\int_E |f|\, dm < \infty.$$

In this case we define

$$\int_E f\, dm = \int_E f^+\, dm - \int_E f^-\, dm.$$

Remark.
A function is integrable exactly when its absolute value has finite integral.

8. Basic Properties of the Integral

Basic Properties of the Integral.

If f and g are integrable on E, and $c \in \mathbb{R}$, then:

1. $$\int_E cf\, dm = c \int_E f\, dm.$$
2. $$\int_E (f + g)\, dm = \int_E f\, dm + \int_E g\, dm.$$
3. If A and B are disjoint measurable sets, then
$$\int_{A \cup B} f\, dm = \int_A f\, dm + \int_B f\, dm.$$

Remark.
These are the usual linearity properties of the integral.

9. Useful Inequalities

Useful Inequalities.

Let f, g be measurable functions.

1. If f is integrable, then
$$\left|\int_E f\,dm\right| \leq \int_E |f|\,dm.$$
2. If $f = g$ almost everywhere and g is integrable, then f is integrable and
$$\int_E f\,dm = \int_E g\,dm.$$
3. If f, g are integrable and $f \leq g$ almost everywhere, then
$$\int_E f\,dm \leq \int_E g\,dm.$$

Remark.
Changing a function on a set of measure zero does not change its integral.

10. Dominated Convergence Theorem

Dominated Convergence Theorem.

Suppose $f_n : E \to \mathbb{R}$ are measurable functions such that:

- $f_n(x) \to f(x)$ almost everywhere,
- there exists an integrable function g such that
$$|f_n(x)| \leq g(x) \quad \text{for all } n.$$

Then f is integrable and
$$\lim_{n\to\infty} \int_E f_n\,dm = \int_E f\,dm.$$

Remark.
If $f_n \to f$ and all f_n are dominated by one integrable function g, then we may pass the limit through the integral.

11. A Simple Special Case

A Simple Special Case.

Let $m(E) < \infty$. Suppose $f_n \to f$ almost everywhere on E, and suppose there is a constant $M > 0$ such that

$$|f_n(x)| \le M \quad \text{for all } x \in E \text{ and all } n.$$

Then

$$\lim_{n\to\infty} \int_E f_n \, dm = \int_E f \, dm.$$

Proof.
Take $g(x) = M$. Since $m(E) < \infty$,

$$\int_E g \, dm = M \, m(E) < \infty.$$

Now apply the Dominated Convergence Theorem.

12. Series and Integration

Series and Integration.

Let $f_1, f_2, f_3, \dots$ be measurable functions on E.
If

$$\sum_{n=1}^{\infty} \int_E |f_n| \, dm < \infty,$$

then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges for almost every x, defines an integrable function F, and

$$\int_E \left(\sum_{n=1}^{\infty} f_n \right) dm = \sum_{n=1}^{\infty} \int_E f_n \, dm.$$

Remark.
Under absolute convergence, we may integrate term by term.

Summary

- First define the integral for simple nonnegative functions.
- Then define the integral of a general nonnegative function by approximation from below.
- The Monotone Convergence Theorem handles increasing sequences.
- Fatou's Lemma gives an important lower-semicontinuity inequality.
- For general real-valued functions, integrability means

$$\int_E |f| \, dm < \infty.$$

- The Dominated Convergence Theorem allows limit and integral to be interchanged under an integrable bound.