

We finished with the first elimination step. To start the second elimination step, since we note that the element $a_{22}^{(1)} = 0$, called the second pivot element, so the simple Gaussian elimination cannot continue in its present form. Therefore, we interchange the rows 2 and 3, to get

$$\begin{pmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 3 \\ 0 & 0 & 1 & \vdots & 1 \end{pmatrix}.$$

We finished with the second elimination step since the element $a_{32}^{(1)}$ is already eliminated from third row. Obviously, the original set of equations has been transformed to an upper-triangular form. Now expressing the set in algebraic form yields

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_2 + 2x_3 &= 3 \\ x_3 &= 1 \end{aligned}$$

Using backward substitution, we get, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, the solution of the system. •

Example 3.12 Use the simple Gaussian elimination method, find all values of a and b for which the following linear system is consistent or inconsistent.

① unique sol. → NO sol.
 ② no sol. ✓

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 1 \\ 4x_1 + 2x_2 + 2x_3 &= 2a \\ 2x_1 + x_2 + x_3 &= b \end{aligned}$$

Solution. Writing the given system in the augmented matrix form

$$\begin{pmatrix} 2 & -1 & 3 & 1 \\ 4 & 2 & 2 & 2a \\ 2 & 1 & 1 & b \end{pmatrix},$$

in which we wish to eliminate the elements a_{21} and a_{31} by subtracting from the second and third rows the appropriate multiples of the first row. In this case the multiples are $m_{21} = 2$ and $m_{31} = 1$. Hence

$$\begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -4 & 2a - 2 \\ 0 & 2 & -2 & b - 1 \end{pmatrix}.$$

We finished with the first elimination step. The second elimination step is to eliminate element $a_{32}^{(1)} = 2$ by subtracting a multiple $m_{32} = \frac{2}{4} = \frac{1}{2}$ of row 2 from row 3, gives

$$\begin{pmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -4 & 2a - 2 \\ 0 & 0 & 0 & b - a \end{pmatrix}.$$

We finished with the second column. So third row of the equivalent upper-triangular system is

$$0x_1 + 0x_2 + 0x_3 = b - a. \quad (3.25)$$

Firstly, if (3.25) has no constraint on unknowns x_1, x_2 , and x_3 , then the upper-triangular system represents only two non-trivial equations, namely

$$\begin{array}{rccccccc} 2x_1 & - & x_2 & + & 3x_3 & = & 1 \\ & & 4x_2 & - & 4x_3 & = & 2a - 2 \end{array}$$

in three unknowns. As a result, one of the unknowns can be chosen arbitrarily, say $x_3 = x_3^*$, then x_2^* and x_1^* can be obtained by using backward substitution

$$x_2^* = a/2 - 1/2 + x_3^*; \quad x_1^* = \frac{1}{2}(1 + a/2 - 1/2 - 2x_3^*).$$

Hence

$$\mathbf{x}^* = [\frac{1}{2}(1/2 + a/2 - 2x_3^*), 1/2a - 1/2 + x_3^*, x_3^*]^T,$$

is an approximation solution of given system for any value of x_3^* for any real value of a . Hence the given linear system is consistent (infinitely many solutions).

Secondly, when $b - a \neq 0$, in this case (3.25) puts a restriction on unknowns x_1, x_2 and x_3 that is impossible to satisfy. So the system cannot have any solutions and therefore, it is inconsistent. •

Example 3.13 For what values of α the following linear system has (i) Unique solution, (ii) No solution, (iii) Infinitely many solutions, by using the simple Gaussian elimination method. Use smallest positive integer value of α to get the unique solution of the system.

$$\begin{array}{rccccccc} x_1 & + & 3x_2 & + & \alpha x_3 & = & 4 \\ 2x_1 & - & x_2 & + & 2\alpha x_3 & = & 1 \\ \alpha x_1 & + & 5x_2 & + & x_3 & = & 6 \end{array}$$

Solution. Writing the given system in the augmented matrix form

$$\left(\begin{array}{cccc} 1 & 3 & \alpha & 4 \\ 2 & -1 & 2\alpha & 1 \\ \alpha & 5 & 1 & 6 \end{array} \right),$$

and by using the following multiples $m_{21} = 2$ and $m_{31} = \alpha$, we get

$$\left(\begin{array}{cccc} 1 & 3 & \alpha & 4 \\ 0 & -7 & 0 & -7 \\ 0 & 5 - 3\alpha & 1 - \alpha^2 & 6 - 4\alpha \end{array} \right).$$

Now using the multiple $m_{32} = \frac{5 - 3\alpha}{-7}$, gives

$$\left(\begin{array}{cccc} 1 & 3 & \alpha & 4 \\ 0 & -7 & 0 & -7 \\ 0 & 0 & 1 - \alpha^2 & 1 - \alpha \end{array} \right).$$

So if $1 - \alpha^2 \neq 0$, then we have the unique solution of the given system while for $\alpha = \pm 1$, we have no unique solution. If $\alpha = 1$, then we have infinitely many solution because third row of above matrix gives

$$0x_1 + 0x_2 + 0x_3 = 0,$$