Example 3.16 *Use the simple Gaussian elimination method to find the inverse of the following matrix*

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -1 & 6 \\ 2 & -3 & 4 \end{pmatrix}.$$

Solution. *Suppose that the inverse $A^{-1} = B$ of the given matrix exists and let*

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -1 & 6 \\ 2 & -3 & 4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}.$$

Now to find the elements of the matrix B, we apply the simple Gaussian elimination on the augmented matrix

$$[A|\mathbf{I}] = \begin{pmatrix} 2 & -1 & 3 & \vdots & 1 & 0 & 0 \\ 4 & -1 & 6 & \vdots & 0 & 1 & 0 \\ 2 & -3 & 4 & \vdots & 0 & 0 & 1 \end{pmatrix}.$$

Applying the forward elimination step of the simple Gaussian elimination on the given matrix A and eliminate the elements $a_{21} = 4$ and $a_{31} = 2$ by subtracting from the second and the third rows the appropriate multiples $m_{21} = \dfrac{4}{2} = 2$ and $m_{31} = \dfrac{2}{2} = 1$ of the first row. It gives

$$\begin{pmatrix} 2 & -1 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & -2 & 1 & 0 \\ 0 & -2 & 1 & \vdots & -1 & 0 & 1 \end{pmatrix}.$$

We finished with the first elimination step. The second elimination step is to eliminate element $a_{32}^{(1)} = -2$ by subtracting a multiple $m_{32} = \dfrac{-2}{1} = -2$ of row 2 from row 3, gives

$$\begin{pmatrix} 2 & -1 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & -2 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -5 & 2 & 1 \end{pmatrix}.$$

We solve the first system

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix},$$

by using backward substitution, we get

$$\begin{array}{rcrcrcr} 2b_{11} & - & b_{21} & + & 3b_{31} & = & 1 \\ & & b_{21} & & & = & -2 \\ & & & & b_{31} & = & -5 \end{array}$$

which gives $b_{11} = 7,\ b_{21} = -2,\ b_{31} = -5$. *Similarly, the solution of the second linear system*

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcrcl} 2b_{12} & - & b_{22} & + & 3b_{32} & = & 0 \\ & & b_{22} & & & = & 1 \\ & & & & b_{32} & = & 2 \end{array}$$

which gives $b_{12} = -5/2,\ b_{22} = 1,\ b_{32} = 2$. *Finally, the solution of the third linear system*

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

can be obtained as follows:

$$\begin{array}{rcrcrcl} 2b_{13} & - & b_{23} & + & 3b_{33} & = & 0 \\ & & b_{23} & & & = & 0 \\ & & & & b_{33} & = & 1 \end{array}$$

and it gives $b_{13} = -3/2,\ b_{23} = 0,\ b_{33} = 1$. *Hence the elements of the inverse matrix* B *are*

$$B = A^{-1} = \begin{pmatrix} 7 & -\frac{5}{2} & -\frac{3}{2} \\ -2 & 1 & 0 \\ -5 & 2 & 1 \end{pmatrix},$$

which is the required inverse of the given matrix A. •

Procedure 3.1 [Gaussian Elimination Method]

1. *Form the augmented matrix,* $B = [A|\mathbf{b}]$.
2. *Check first pivot element* $a_{11} \neq 0$*, then move to the next step; otherwise, interchange rows so that* $a_{11} \neq 0$.
3. *Multiply row one by multiplier* $m_{i1} = \dfrac{a_{i1}}{a_{11}}$ *and subtract to the ith row for* $i = 2, 3, \ldots, n$.
4. *Repeat the steps 2 and 3 for the remaining pivots elements unless coefficient matrix* A *becomes upper-triangular matrix* U.
5. *Use backward substitution to solve* x_n *from the nth equation* $x_n = \dfrac{b_n^{n-1}}{a_{nn}}$ *and solve the other (n-1) unknowns variables by using (3.21).*

The use of non-zero pivots is sufficient for the theoretical correctness of the simple Gaussian elimination, but more care must be taken if one is to obtain reliable results.