Coupled Oscillations

Definition:

- linear chain of n identical bodies (mass m) connected to one another and to fixed endpoints by identical ideal springs (spring constant k)
- distances from equilibrium x_i, $i=1 \dots n$
- zero initial velocities; friction ignored

Many important physics systems involved coupled oscillators. Coupled oscillators are oscillators connected in such a way that energy can be transferred between them. The motion of coupled oscillators can be complex, and does not have to be periodic. However, when the oscillators carry out complex motion, we can find a coordinate frame in which each oscillator oscillates with a very well defined frequency **(normal coordinates)**

A solid is a good example of a system that can be described in terms of coupled oscillations. The atoms oscillate around their equilibrium positions, and the interaction between the atoms is responsible for the coupling. To start our study of coupled oscillations, we will assume that the forces involved are spring-like forces (the magnitude of the force is proportional to the magnitude of the displacement from equilibrium).

Two Coupled Harmonic Oscillators

Consider a system of two objects of mass M. The two objects are attached to two springs with spring constants κ (see Figure 1). The interaction force between the masses is represented by a third spring with spring constant κ_{12}, which connects the two masses.

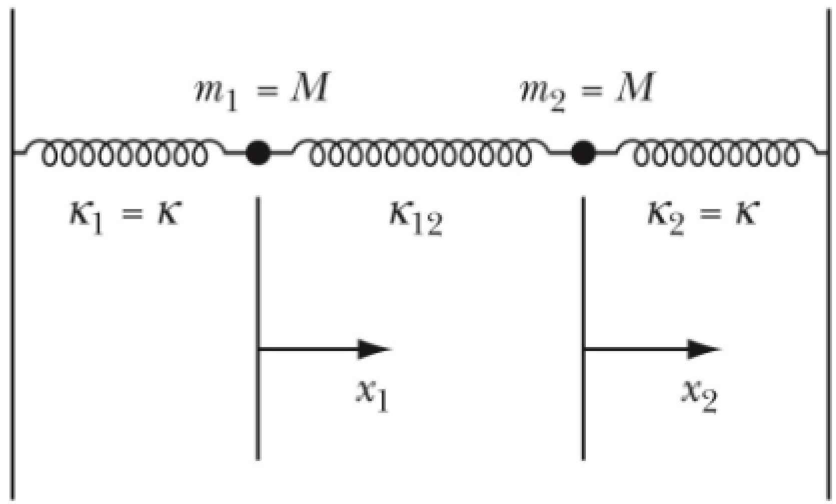

Figure 1. Two coupled harmonic oscillators.

We will assume that when the masses are in their equilibrium position, the springs are also in their equilibrium positions. The force on the left mass is equal to

We will assume that when the masses are in their equilibrium position, the springs are also in their equilibrium positions. The force on the left mass is equal to

$$F_1 = -\kappa x_1 + \kappa_{12}\left(x_2 - x_1\right) = -\left(\kappa + \kappa_{12}\right)x_1 + \kappa_{12}x_2 = M\ddot{x}_1$$

The force on the right mass is equal to

$$F_2 = -\kappa x_2 + \kappa_{12}\left(x_1 - x_2\right) = -\left(\kappa + \kappa_{12}\right)x_2 + \kappa_{12}x_1 = M\ddot{x}_2$$

The equations of motion are thus

$$\left.\begin{aligned} M\ddot{x}_1 + \left(\kappa + \kappa_{12}\right)x_1 - \kappa_{12}x_2 &= 0 \\ M\ddot{x}_2 + \left(\kappa + \kappa_{12}\right)x_2 - \kappa_{12}x_1 &= 0 \end{aligned}\right\} \quad (12.1)$$

Since it is reasonable to assume that the resulting motion has an oscillatory behavior, we consider following trial functions:

$$x_1(t) = B_1 e^{i\omega t}$$

$$x_2(t) = B_2 e^{i\omega t}$$

Substituting these trial functions into the equations of motion we obtain the following conditions:

$$\left(\kappa + \kappa_{12} - M\omega^2\right) B_1 - \kappa_{12} B_2 = 0$$

$$-\kappa_{12} B_1 + \left(\kappa + \kappa_{12} - M\omega^2\right) B_2 = 0$$

These equations only will have a non-trivial solution if

$$\begin{vmatrix} \kappa + \kappa_{12} - M\omega^2 & -\kappa_{12} \\ -\kappa_{12} & \kappa + \kappa_{12} - M\omega^2 \end{vmatrix} = 0$$

Note: the trivial solution is $B_1 = B_2 = 0$. The requirement for a non-trivial solution requires that the angular frequency of the system is equal to one of the following two characteristic frequencies (the so called eigen frequencies):

$$\omega_1 = \pm\sqrt{\frac{\kappa + 2\kappa_{12}}{M}}$$

$$\omega_2 = \pm\sqrt{\frac{\kappa}{M}}$$

For each of these frequencies, we can now determine the amplitudes B_1 and B_2. Let us first consider the eigen frequency ω_1. For this frequency we obtain the following relations between B_1 and B_2:

$$\left(\kappa + \kappa_{12} - \left(\kappa + 2\kappa_{12}\right)\right)B_1 - \kappa_{12}B_2 = -\kappa_{12}B_1 - \kappa_{12}B_2 = -\kappa_{12}\left(B_1 + B_2\right) = 0$$

or B_1 = -B_2. For the eigen frequency ω_2 we obtain the following relations between B_1 and B_2:

$$(\kappa+\kappa_{12}-\kappa)B_1-\kappa_{12}B_2=\kappa_{12}B_1-\kappa_{12}B_2=\kappa_{12}\left(B_1-B_2\right)=0$$

or $B_1 = B_2$. The most general solution of the coupled harmonic oscillator problem is thus

$$x_1(t)=B_1^{\ +}e^{+i\omega_1 t}+B_1^{\ -}e^{-i\omega_1 t}+B_2^{\ +}e^{+i\omega_2 t}+B_2^{\ -}e^{-i\omega_2 t}$$

$$x_2(t)=-B_1^{\ +}e^{+i\omega_1 t}-B_1^{\ -}e^{-i\omega_1 t}+B_2^{\ +}e^{+i\omega_2 t}+B_2^{\ -}e^{-i\omega_2 t}$$

Another approach that can be used to solve the coupled harmonic oscillator problem is to carry out a coordinate transformation that decouples the coupled equations. Consider the two equations of motion. If we add them together we get

$$M\left(\ddot{x}_1+\ddot{x}_2\right)+\kappa\left(x_1+x_2\right)=0$$

If we subtract from each other we get

$$M(\ddot{x}_1-\ddot{x}_2)+(\kappa+2\kappa_{12})(x_1-x_2)=0$$

Based on these two equations it is clear that in order to decouple the equations of motion we need to introduce the following variables

$$\eta_1 = x_1 - x_2$$

$$\eta_2 = x_1 + x_2$$

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(\eta_2+\eta_1) \\ x_2 &= \frac{1}{2}(\eta_2-\eta_1) \end{aligned}\right\}$$

Substituting these expressions for x_1 and x_2 into Equation 12.1, we find

$$\left.\begin{aligned} M(\ddot{\eta}_1 + \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 + \kappa\eta_2 = 0 \\ M(\ddot{\eta}_1 - \ddot{\eta}_2) + (\kappa + 2\kappa_{12})\eta_1 - \kappa\eta_2 = 0 \end{aligned}\right\}$$

which can be solved (by adding and subtracting) to yield

$$\left.\begin{aligned} M\ddot{\eta}_1 + (\kappa + 2\kappa_{12})\eta_1 = 0 \\ M\ddot{\eta}_2 + \kappa\eta_2 = 0 \end{aligned}\right\}$$

The solutions to the decoupled equations of motion are

$$\eta_1(t) = C_1^{+}e^{i\omega_1 t} + C_1^{-}e^{-i\omega_1 t}$$

$$\eta_2(t) = C_2^{+}e^{i\omega_2 t} + C_2^{-}e^{-i\omega_2 t}$$

We note that the solution η_1 corresponds to an asymmetric motion of the masses, while the solution η_2 corresponds to an asymmetric motion of the masses (see Figure 2). Since higher frequencies correspond to higher energies, the asymmetric mode (out of phase) has a higher energy.

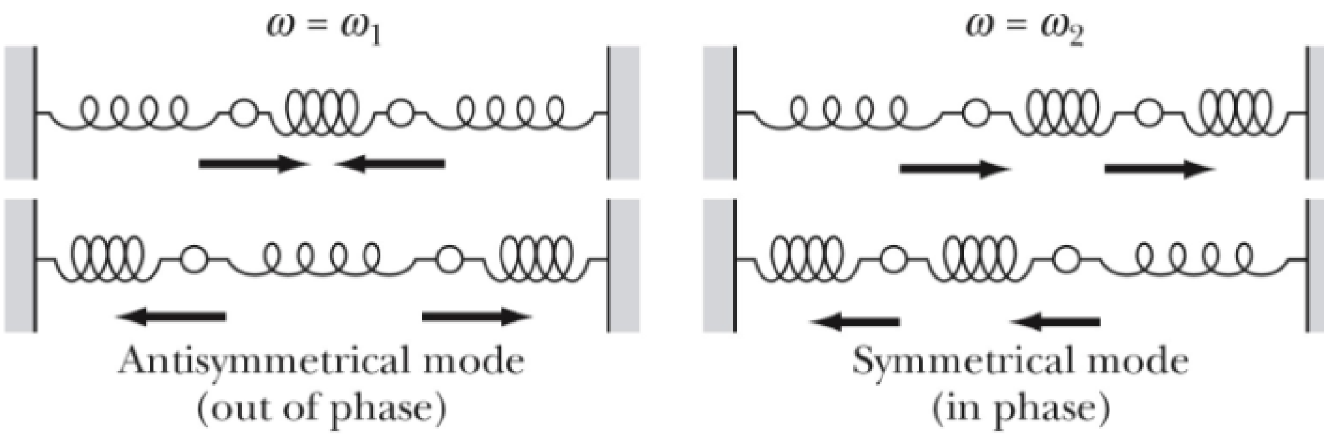

Figure 2. Normal modes of oscillation.