Motion in a Noninertial Reference Frame

Ch. 10

10.1 Introduction

an *absolute inertial* frame, i.e., a frame that is at absolute rest and one in which Newton's laws are absolutely valid.

Experience has shown that, if relativistic effects can be neglected, the motion of a particle in an inertial reference frame is correctly described by the Newtonian equation F = p. In the event that the particle is not required to move in some complicated manner and if rectangular coordinates are used to describe the motion, then usually the equations of motion are relatively simple. However, if either of these restrictions is removed, the equations can become quite complex and difficult to manipulate.

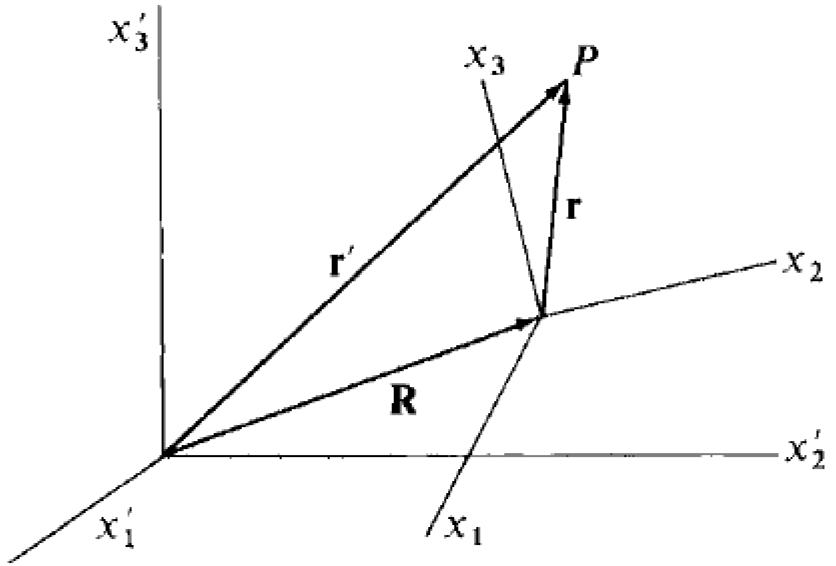
In order to describe, for example, the motion of a particle on or near the surface of the Earth, it is clearly tempting to do so by choosing a coordinate system fixed with respect to the Earth. We know, however, that the Earth undergoes a complicated motion, compounded of many different rotations (and, hence, accelerations) with respect to an inertial reference frame identified with the "fixed" stars. The Earth coordinate system is, therefore, a *noninertial frame of reference*

10.2 Rotating Coordinate Systems

Let us consider two sets of coordinate axes; let one set be the "fixed" or inertial axes, and let the other be in motion with respect to the inertial system. We shall designate these axes as the "fixed" and "rotating" axes, respectively, and shall use x_i' *as coordinates* in the fixed system and x_i as coordinates in the rotating system.

If we choose some point P, as in Fig. , we clearly have

$$\mathbf{r}' = \mathbf{R} + \mathbf{r}$$



where **r'** is the radius vector of ***P** in the fixed system and where **r** is the* radius vector of ***P** in the rotating system. The vector **R** locates the origin* of the rotating system in the fixed system.

if the $\mathbf{x}_i$ system undergoes an infinitesimal rotation $\delta\theta$, corresponding to some arbitrary infinitesimal displacement, the motion of *P (which, for the moment, we consider to be* at rest in the $\mathbf{x}_i$ system) may be described as

$$(d\mathbf{r})_{\text{fixed}} = d\boldsymbol{\theta} \times \mathbf{r} \quad \longrightarrow \quad \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \frac{d\boldsymbol{\theta}}{dt} \times \mathbf{r}$$

$$\boldsymbol{\omega} \equiv \frac{d\boldsymbol{\theta}}{dt} \quad \longrightarrow \quad \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \boldsymbol{\omega} \times \mathbf{r} \qquad (\text{for } P \text{ fixed in } x_i \text{ system})$$

Now, if we allow the point ***P*** *to have velocity* $\left(\mathbf{dr/dt}\right)_{rotating}$ *with respect* to the $\mathbf{x}_i$ system, this velocity must be added to **ω x r** to obtain the time rate of change of **r** in the fixed system :

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r}$$

In fact, for an arbitrary vector Q, we have

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{Q}$$

We note, for example, that the angular acceleration $\dot{\boldsymbol{\omega}}$ is the same in both the fixed and rotating systems:

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \boldsymbol{\omega} \equiv \dot{\boldsymbol{\omega}}$$

since $\boldsymbol{\omega} \times \boldsymbol{\omega}$ vanishes and where $\dot{\boldsymbol{\omega}}$ designates the common value in the two systems.

Equation may now be used to obtain the expression for the velocity of the point *P as measured in the fixed coordinate system.*

$$\left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}}$$

so that

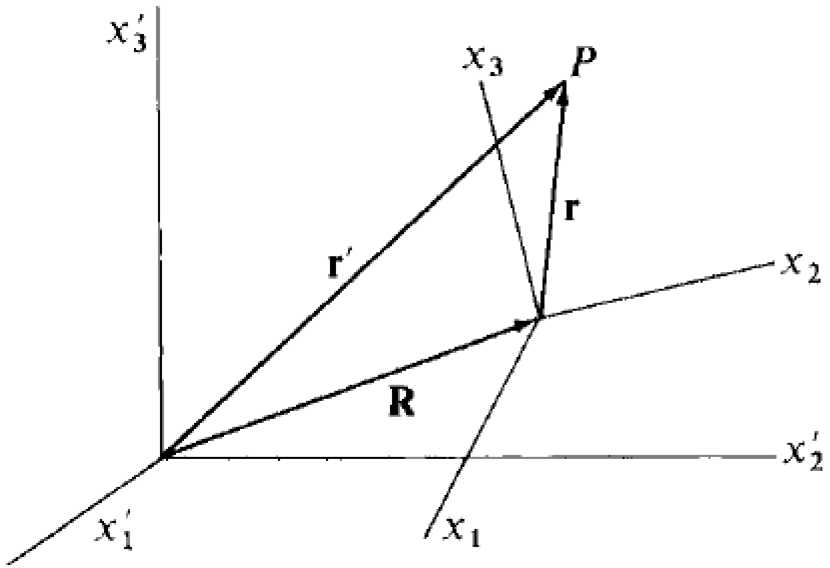
$$\left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{r}$$

If we define

$$\mathbf{v}_f \equiv \dot{\mathbf{r}}_f \equiv \left(\frac{d\mathbf{r}'}{dt}\right)_{\text{fixed}}, \quad \mathbf{V} \equiv \dot{\mathbf{R}}_f \equiv \left(\frac{d\mathbf{R}}{dt}\right)_{\text{fixed}}, \quad \mathbf{v}_r \equiv \dot{\mathbf{r}}_r \equiv \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}}$$

we may write

$$\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$$



where

$\mathbf{v}_f$ = velocity relative to the fixed axes

$\mathbf{V}$ = linear velocity of the moving origin

$\mathbf{v}_r$ = velocity relative to the rotating axes

$\boldsymbol{\omega}$ = angular velocity of the rotating axes

$\boldsymbol{\omega} \times \mathbf{r}$ = velocity due to the rotation of the moving axes

EXAMPLE 10.1

Consider a vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ in the rotating system. Let the fixed and rotating systems have the same origin. Find $\dot{\mathbf{r}}'$ in the fixed system by direct differentiation if the angular velocity of the rotating system is $\boldsymbol{\omega}$ in the fixed system.

Solution. We begin by taking the time derivative directly

$$\begin{aligned}\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} &= \frac{d}{dt}\left(\sum_{\mathrm{i}} \mathrm{x_i}\, \mathbf{e}_i\right) \\ &= \sum_i (\dot{x}_i\mathbf{e}_i + x_i\dot{\mathbf{e}}_i)\end{aligned} \qquad \textbf{(10.7)}$$

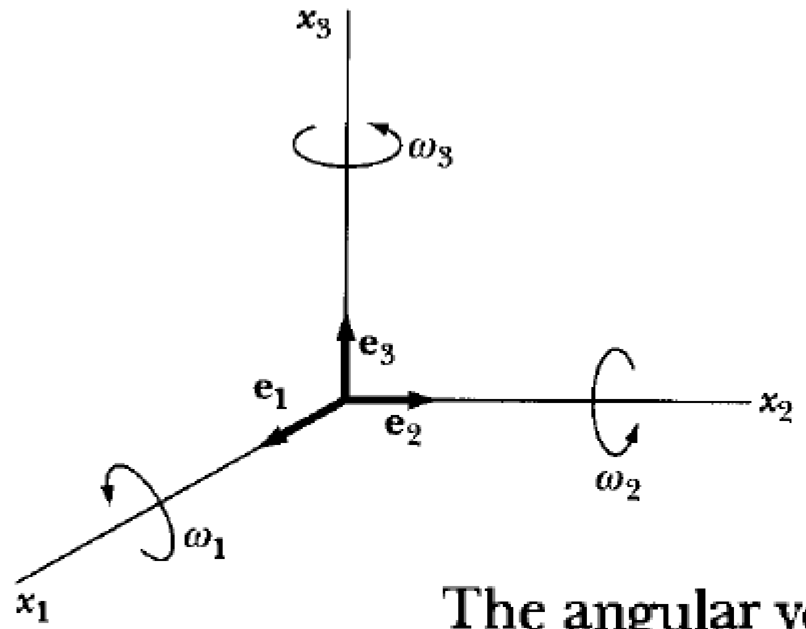
The first term is simply $\dot{\mathbf{r}}_r$ in the rotating system, but what are the $\dot{\mathbf{e}}_i$?

$$\dot{\mathbf{r}}_r = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}}$$

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \dot{\mathbf{r}}_r + \sum_i x_i \dot{\mathbf{e}}_i \tag{10.8}$$

Look at Figure 10-2 and examine which components of ω_i tend to rotate $\mathbf{e}_1$. We see that ω_2 tends to rotate $\mathbf{e}_1$ toward the $-\mathbf{e}_3$ direction and that ω_3 tends to rotate $\mathbf{e}_1$ toward the $+\mathbf{e}_2$ direction. We therefore have

$$\frac{d\mathbf{e}_1}{dt} = \omega_3\mathbf{e}_2 - \omega_2\mathbf{e}_3 \tag{10.9a}$$



The angular velocity components ω_i rotate the system around the $\mathbf{e}_i$ axis, so that, for example, ω_3 tends to rotate $\mathbf{e}_1$ toward the $+\mathbf{e}_2$ direction.

Similarly, we have

$$\frac{d\mathbf{e}_2}{dt} = -\omega_3\mathbf{e}_1 + \omega_1\mathbf{e}_3 \tag{10.9b}$$

$$\frac{d\mathbf{e}_3}{dt} = \omega_2\mathbf{e}_1 - \omega_1\mathbf{e}_2 \tag{10.9c}$$

In each case, the direction of the time derivative of the unit vector must be perpendicular to the unit vector in order not to change its magnitude.

Equations 10.9a–c can be written

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \tag{10.10}$$

and Equation 10.8 becomes

$$\begin{aligned}\left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} &= \dot{\mathbf{r}}_r + \sum_i \boldsymbol{\omega} \times x_i\mathbf{e}_i \\ &= \dot{\mathbf{r}}_r + \boldsymbol{\omega} \times \mathbf{r} \end{aligned} \tag{10.11}$$

10.3 Centrifugal and Coriolis force

We have seen that Newton's equation F = ma is valid only in an inertial frame of reference. Therefore, the expression for the force on a particle may be obtained from

$$\mathbf{F} = m\mathbf{a}_f = m\left(\frac{d\mathbf{v}_f}{dt}\right)_{\text{fixed}}$$

where the differentiation must be carried out with respect to the fixed system.

we have $$\boxed{\mathbf{v}_f = \mathbf{V} + \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}}$$

Differentiating the last equation , we have

$$\left(\frac{d\mathbf{v}_f}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{V}}{dt}\right)_{\text{fixed}} + \left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{fixed}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} \quad \Rightarrow \quad (1)$$

We denote the first term by

$$\ddot{\mathbf{R}}_f \equiv \left(\frac{d\mathbf{V}}{dt}\right)_{\text{fixed}}$$

The second term may be evaluated by substituting v_r for Q in the following equation from the previous section:

$$\boxed{\left(\frac{d\mathbf{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{Q}}$$

So the second term becomes,

$$\left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{v}_r}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{v}_r$$

$$= \mathbf{a}_r + \boldsymbol{\omega} \times \mathbf{v}_r$$

where $\mathbf{a}_r$, is the acceleration in the rotating coordinate system.

The last term may be obtained directly

$$\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{fixed}} = \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= \boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Substituting these terms into equation (1) , we have

$$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

To an observer in the rotating coordinate system, the effective force on a particle is given by

$$\mathbf{F}_{\text{eff}} = m\mathbf{a}_r = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r \quad \rightarrow \quad (2)$$

$$\mathbf{F}_{\text{eff}} \equiv m\mathbf{a}_r = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r$$

The first term, **F**, is the sum of the forces acting on the particle as measured in the fixed inertial system. The second ($-m\ddot{\mathbf{R}}_f$) and third ($-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$) terms result because of the translational and angular acceleration, respectively, of the moving coordinate system relative to the fixed system.

The quantity $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the usual *centrifugal force* term and reduces to $m\omega^2 r$ for the case in which $\boldsymbol{\omega}$ is normal to the radius vector. Note that the minus sign implies that the centrifugal force is directed *outward* from the center of rotation

The last term in Equation 10.25 is a totally new quantity that arises from the motion of the particle in the rotating coordinate system. This term is called the **Coriolis force.** Note that the Coriolis force does indeed arise from the *motion* of the particle, because the force is proportional to v_r and hence vanishes if there is no motion.

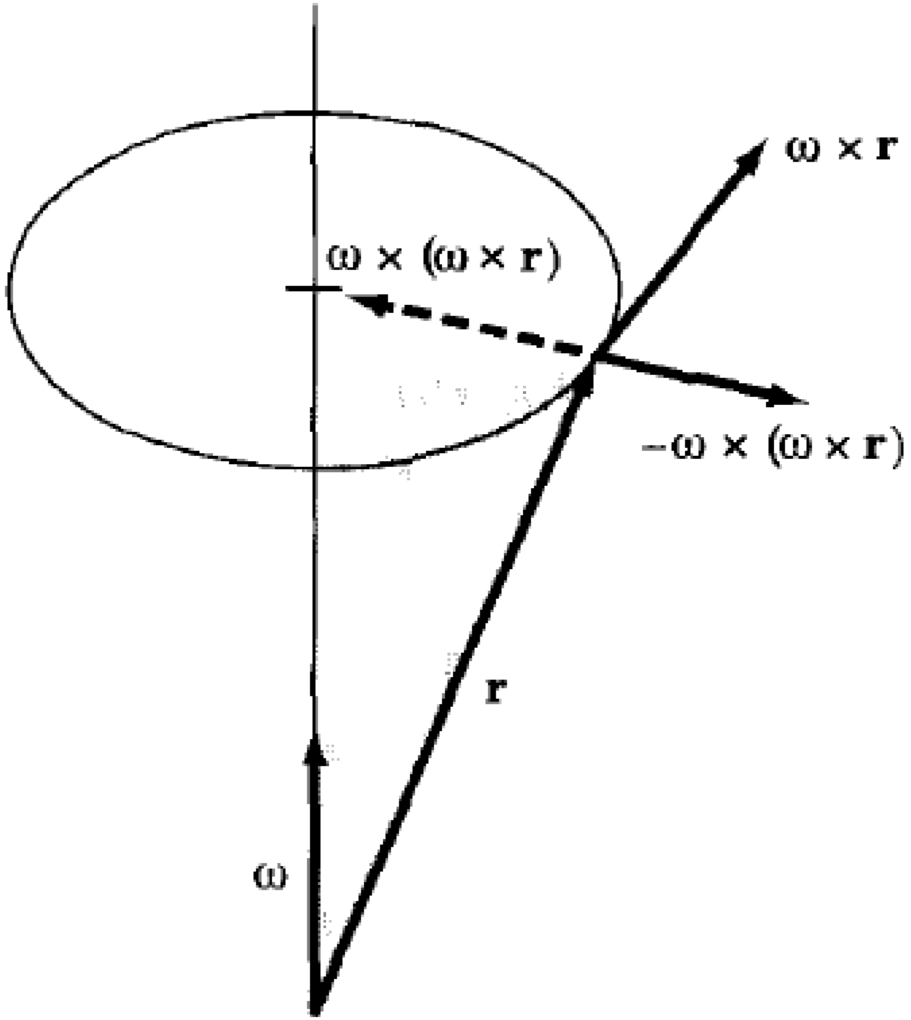


Diagram indicating that the vector $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ points outward, away from the axis of rotation along $\boldsymbol{\omega}$. The term $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the usual centrifugal force.

In eq. (2) (let $\ddot{\mathbf{R}}_f'$ and $\dot{\boldsymbol{\omega}}$ be zero for simplicity) , we get

$$\mathbf{F}_{\text{eff}} = m\mathbf{a}_f + \text{(noninertial terms)}$$

where the "noninertial terms" are identified as the centrifugal and Coriolis "forces." Thus, for example, if a body rotates about a fixed force center, the only real force on the body is the force of attraction toward the force center (and gives rise to the *centripetal* acceleration). An observer moving with the rotating body, however, measures this central force and also notes that the body does not fall toward the force center. To reconcile this result with the requirement that the net force on the body vanish, the observer must postulate an additional force—the centrifugal force. But the "requirement" is artificial; it arises solely from an attempt to extend the form of Newton's equation to a noninertial system, and this can be done only by introducing a fictitious "correction force." The same comments apply for the Coriolis force; this "force" arises when an attempt is made to describe motion relative to the rotating body.

EXAMPLE 10.2

A student is performing measurements with a hockey puck on a large merry-go-round with a smooth (frictionless) horizontal, flat surface. The merry-go-round has a constant angular velocity $\boldsymbol{\omega}$ and rotates counterclockwise as seen from above. (a) Find the effective force on the hockey puck after it is given a push. (b) Plot the path for various initial directions and velocities of the puck as observed by the person on the merry-go-round that pushes the puck.

Solution. The first three terms for $\mathbf{F}_{\text{eff}}$ in Equation 10.25 are zero, so the effective force as observed by the person on the merry-go-round is

$$\mathbf{F}_{\text{eff}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r \tag{10.26}$$

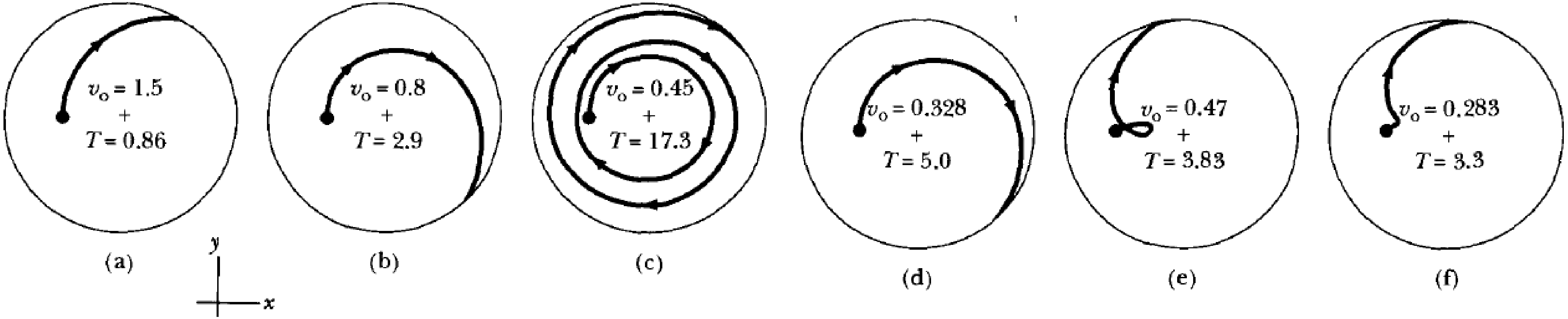
We have taken the frictional force to be zero. Remember that $\mathbf{v}_r$ is the velocity as measured by the observer on the rotating surface. The effective acceleration is

$$\mathbf{a}_{\text{eff}} = \frac{\mathbf{F}_{\text{eff}}}{m} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \mathbf{v}_r \tag{10.27}$$

The velocity and position are given by integration, in turn, of the acceleration.

$$\mathbf{v}_{\text{eff}} = \int \mathbf{a}_{\text{eff}}\, dt \qquad \mathbf{r}_{\text{eff}} = \int \mathbf{v}_{\text{eff}}\, dt \tag{10.28}$$

We put the origin of our rotating coordinate system at the center of the merry-go-round. We will need the initial positions and velocities of the puck to plot the motion. For this example, we let the radius of the merry-go-round be R and the velocities be in units of ωR. The initial position of the puck will always be at an (x, y) position of $(-0.5R, 0)$.



The motion of the hockey puck of Example 10.2 as observed in the rotating system for various initial directions and velocities v_0 at the times T noted. The angular velocity ω(1 rad/s) is out of the page.

In each of the cases above, the puck will move in a straight line in the fixed system, because there is no friction or external force in the plane.

10.3 Motion Relative to the Earth

The motion of Earth with respect to an inertial reference frame is dominated by Earth's rotation about its own axis. The effects of the other motions (e.g., the revolution about the Sun and the motion of the solar system with respect to the local galaxy) are small by comparison. If we place the fixed inertial frame $x'y'z'$ at the center of Earth and the moving reference frame xyz on the surface of Earth, we can describe the motion of a moving object close to the surface of Earth as shown in Figure . We denote the forces as measured in the fixed inertial system as $\mathbf{F} = \mathbf{S} + m\mathbf{g}_0$, where $\mathbf{S}$ represents the sum of the external forces other than gravitation, and $m\mathbf{g}_0$ represents the gravitational attraction to Earth.

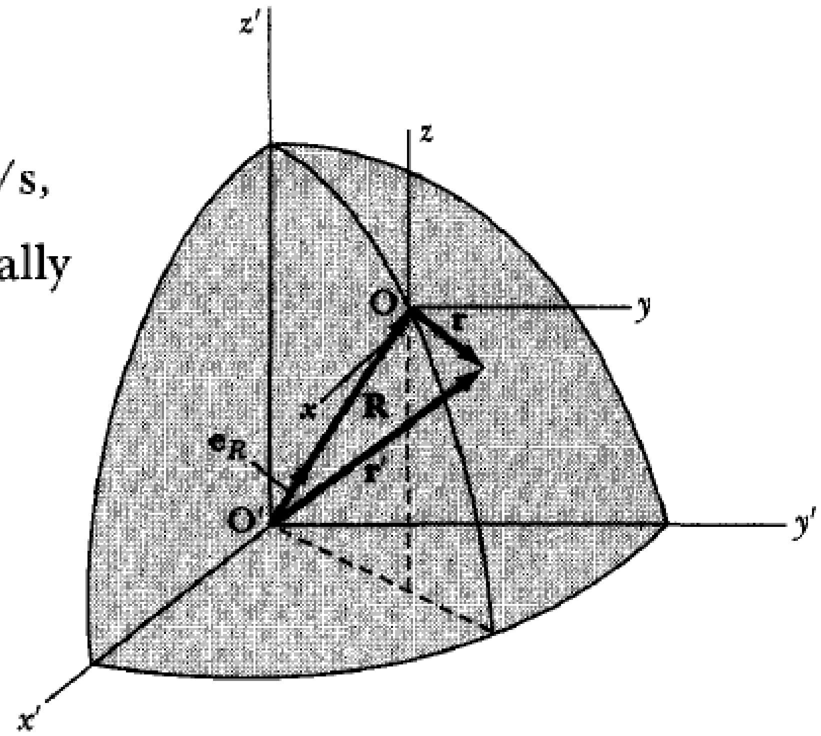
$$\mathbf{g}_0 = -G\frac{M_E}{R^2}\mathbf{e}_R$$

where M_E is the mass of Earth, R is the radius of Earth, and the unit vector $\mathbf{e}_R$ is a unit vector along the direction of $\mathbf{R}$

The effective force $\mathbf{F}_{\text{eff}}$ as measured in the moving system placed on the surface of Earth becomes,

$$\mathbf{F}_{\text{eff}} = \mathbf{S} + m\mathbf{g}_0 - m\ddot{\mathbf{R}}_f - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_r \qquad (10.30)$$

We let Earth's angular velocity $\boldsymbol{\omega}$ be along the inertial system's z'-direction ($\mathbf{e}'_z$). The value of ω is 7.3×10^{-5} rad/s, which is a relatively slow rotation, The value of $\boldsymbol{\omega}$ is practically constant in time, and the term $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ will be neglected.



In order to study the motion of an object near Earth's surface, we place a fixed inertial frame $x'y'z'$ at the center of Earth and the moving frame xyz on Earth's surface.

According to Equation $\left(\frac{d\mathbf{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\text{rotating}} + \boldsymbol{\omega} \times \mathbf{Q}$

we have for the third term above,

$$\ddot{\mathbf{R}}_f = \boldsymbol{\omega} \times \dot{\mathbf{R}}_f$$

$$\ddot{\mathbf{R}}_f = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) \tag{10.31}$$

Equation 10.30 now becomes

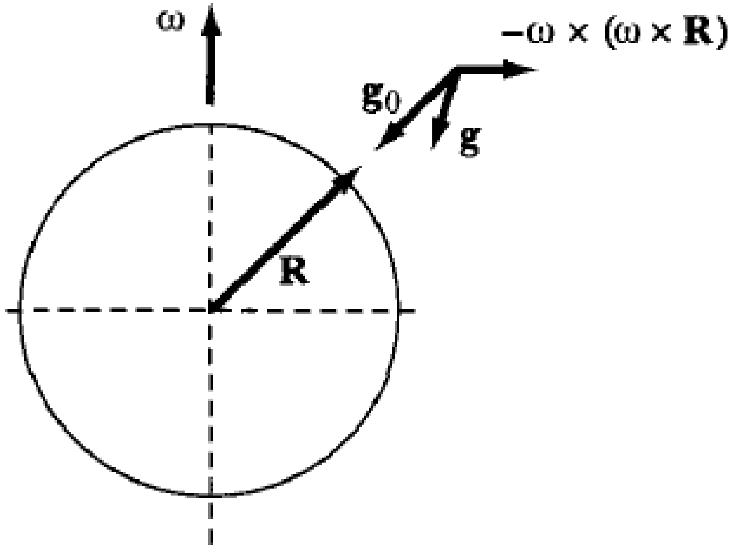
$$\mathbf{F}_{\text{eff}} = \mathbf{S} + m\mathbf{g}_0 - m\boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R})] - 2m\boldsymbol{\omega} \times \mathbf{v}_r \tag{10.32}$$

The second and third terms (divided by m) are what we experience (and measure) on the surface of Earth as the effective $\mathbf{g}$,

$$\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R})] \tag{10.33}$$

The second term of Equation 10.33 is the centrifugal force. Because we are limiting our present consideration to motion near the surface of Earth, we have $r \ll R$, and the $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ term totally dominates the centrifugal force.

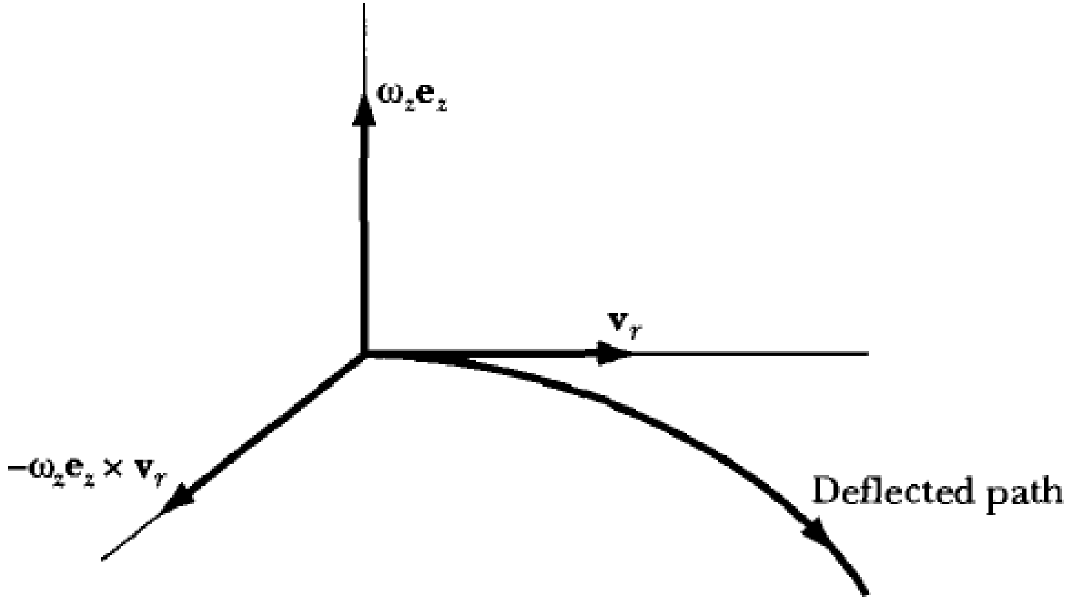
The value of $\omega^2 R$ is 0.034 m/s^2, and this is a significant enough amount (0.35%) of the magnitude of $\mathbf{g}$ to be considered. The direction of the centrifugal term $(-\boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{R})]$ is outward from the axis of the rotating Earth. The direction of a plumb bob will include the centrifugal term. Because of this fact, the direction of $\mathbf{g}$ at a given point is in general slightly different from the true vertical



Near Earth's surface the terms $\mathbf{g}_0$ (Earth's gravitational field vector) and $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ (main centrifugal term) make up the effective $\mathbf{g}$ (other smaller terms have been neglected). The effect of the centrifugal term on $\mathbf{g}$ is exaggerated here.

Coriolis Force Effects

The angular velocity vector $\boldsymbol{\omega}$, which represents Earth's rotation about its axis, is directed in a northerly direction. Therefore, in the Northern Hemisphere, $\boldsymbol{\omega}$ has a component ω_z directed *outward* along the local vertical. If a particle is projected in a horizontal plane (in the local coordinate system at the surface of Earth) with a velocity $\mathbf{v}_r$, then the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}_r$ has a component in the plane of magnitude $2m\omega_z v_r$ directed toward the *right* of the particle's motion (see Figure), and a deflection from the original direction of motion results.



In the Northern Hemisphere, a particle projected in a horizontal plane will be directed toward the right of the particle's motion. In the Southern Hemisphere, the direction will be to the left.

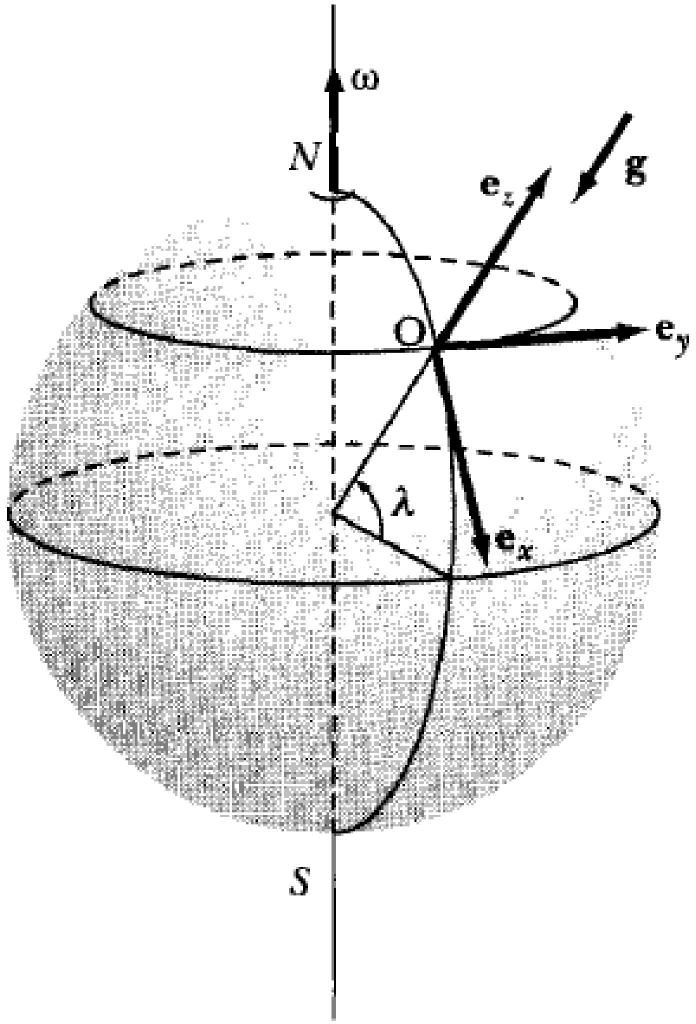
EXAMPLE 10.3

Find the horizontal deflection from the plumb line caused by the Coriolis force acting on a particle falling freely in Earth's gravitational field from a height h above Earth's surface.

Solution. We use Equation 10.34 with the applied forces $\mathbf{S} = 0$. If we set $\mathbf{F}_{\text{eff}} = m\mathbf{a}_r$, we can solve for the acceleration of the particle in the rotating coordinate system fixed on Earth.

$$\mathbf{a}_r = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{v}_r$$

The acceleration due to gravity $\mathbf{g}$ is the effective one and is along the plumb line. We choose a z-axis directed vertically outward (along $-\mathbf{g}$) from the surface of



The coordinate system on Earth's surface for finding the horizontal deflection of a falling particle from the plumb line caused by the Coriolis force. The vector $\mathbf{e}_x$ is in the southerly direction, and $\mathbf{e}_y$ is in the easterly direction.

Because we have chosen the origin O of the rotating coordinate system to lie in the Northern Hemisphere, we have

$$\omega_x = -\omega \cos \lambda$$

$$\omega_y = 0$$

$$\omega_z = \omega \sin \lambda$$

Although the Coriolis force produces small velocity components in the $\mathbf{e}_y$ and $\mathbf{e}_x$ directions, we can certainly neglect $\dot{x}$ and $\dot{y}$ compared with $\dot{z}$, the vertical velocity. Then, approximately,

$$\dot{x} \cong 0$$

$$\dot{y} \cong 0$$

$$\dot{z} \cong -gt$$

where we obtain $\dot{z}$ by considering a fall from rest. Therefore, we have

$$\boldsymbol{\omega} \times \mathbf{v}_r \cong \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\omega \cos \lambda & 0 & \omega \sin \lambda \\ 0 & 0 & -gt \end{vmatrix}$$

$$\cong -(\omega g t \cos \lambda)\mathbf{e}_y$$

The components of $\mathbf{g}$ are

$$g_x = 0$$

$$g_y = 0$$

$$g_z = -g$$

so the equations for the components of $\mathbf{a}_r$ (neglecting terms* in ω^2; become

$$(\mathbf{a}_r)_x = \ddot{x} \cong 0$$

$$(\mathbf{a}_r)_y = \ddot{y} \cong 2\omega g t \cos \lambda$$

$$(\mathbf{a}_r)_z = \ddot{z} \cong -g$$

Thus, the effect of the Coriolis force is to produce an acceleration in the $\mathbf{e}_y$, or easterly, direction. Integrating $\ddot{y}$ twice, we have

$$y(t) \cong \frac{1}{3}\omega g t^3 \cos \lambda$$

where $y = 0$ and $\dot{y} = 0$ at $t = 0$. The integration of $\ddot{z}$ yields the familiar result for the distance of fall,

$$z(t) \cong z(0) - \frac{1}{2}gt^2$$

and the time of fall from a height $h = z(0)$ is given by

$$t \cong \sqrt{2h/g}$$

Hence the result for the eastward deflection d of a particle dropped from rest at a height h and at a northern latitude λ is†

$$d \cong \frac{1}{3}\omega \cos \lambda \sqrt{\frac{8h^3}{g}}$$

An object dropped from a height of 100 m at latitude 45° is deflected approximately 1.55 cm (neglecting the effects of air resistance).

EXAMPLE 10.5

The effect of the Coriolis force on the motion of a pendulum produces a *precession*, or rotation with time of the plane of oscillation. Describe the motion of this system, called a *Foucault pendulum.**

Solution. To describe this effect, let us select a set of coordinate axes with origin at the equilibrium point of the pendulum and z-axis along the local vertical. We are interested only in the rotation of the plane of oscillation—that is, we wish to consider the motion of the pendulum bob in the x–y plane (the horizontal plane). We therefore limit the motion to oscillations of small amplitude, with the horizontal excursions small compared with the length of the pendulum. Under this condition, $\dot{z}$ is small compared with $\dot{x}$ and $\dot{y}$ and can be neglected.

The equation of motion is

$$\mathbf{a}_r = \mathbf{g} + \frac{\mathbf{T}}{m} - 2\boldsymbol{\omega} \times \mathbf{v}_r \tag{10.42}$$

where $\mathbf{T}/m$ is the acceleration produced by the force of tension $\mathbf{T}$ in the pendulum suspension (Figure 10-11). We therefore have, approximately,

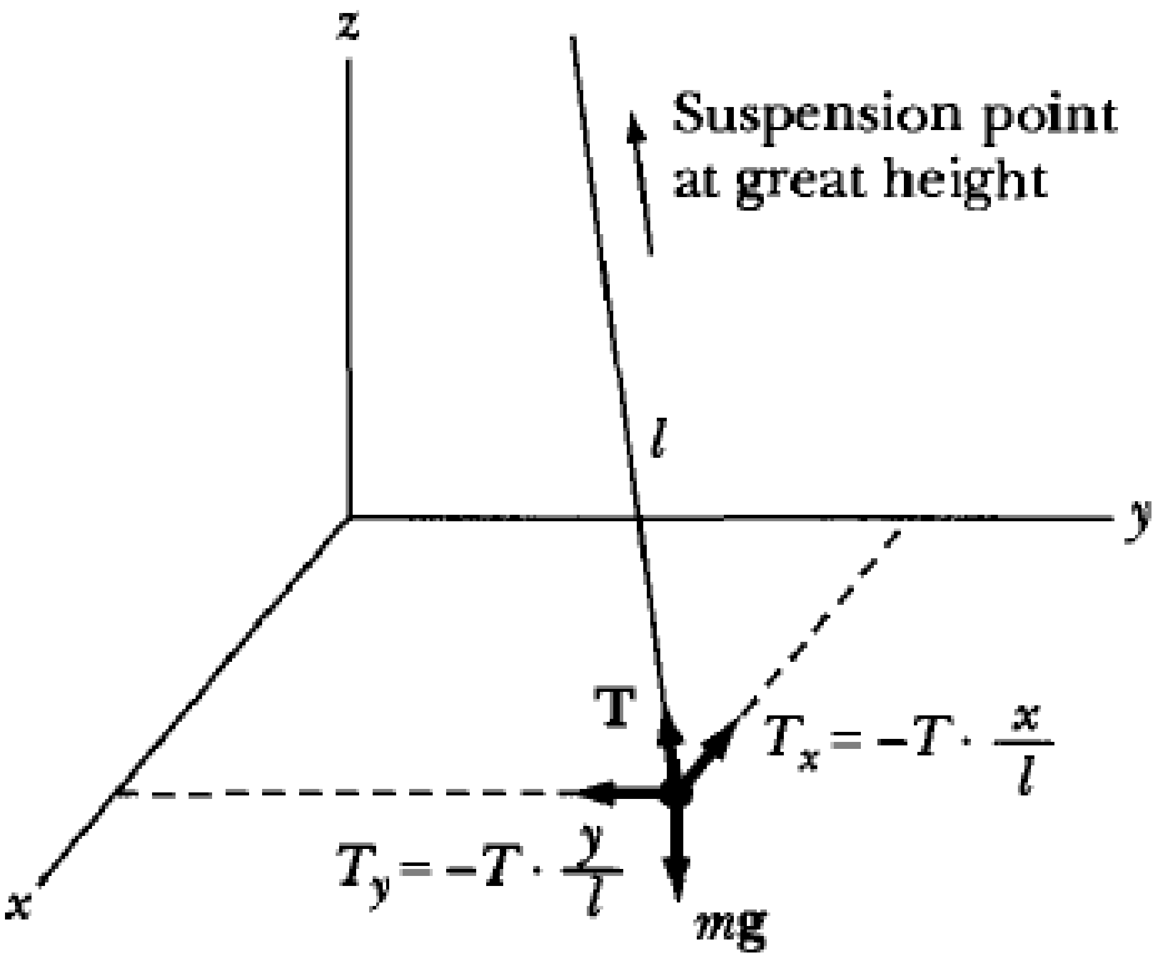
$$\left.\begin{aligned} T_x &= -T\cdot\frac{x}{l} \\ T_y &= -T\cdot\frac{y}{l} \\ T_z &\cong T \end{aligned}\right\} \tag{10.43}$$

As before,

$$\boxed{\begin{aligned} g_x &= 0 \\ g_y &= 0 \\ g_z &= -g \end{aligned}}$$

and

$$\boxed{\begin{aligned} \omega_x &= -\omega\cos\lambda \\ \omega_y &= 0 \\ \omega_z &= \omega\sin\lambda \end{aligned}}$$



Geometry for the Foucault pendulum. The acceleration **g** vector is along the $-z$-direction, and the tension **T** is separated into x-, y-, and z-components.

with

$$(\mathbf{v}_r)_x = \dot{x}$$

$$(\mathbf{v}_r)_y = \dot{y}$$

$$(\mathbf{v}_r)_z = \dot{z} \cong 0$$

Therefore,

$$\boldsymbol{\omega} \times \mathbf{v} \cong \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -\omega \cos \lambda & 0 & \omega \sin \lambda \\ \dot{x} & \dot{y} & 0 \end{vmatrix}$$

so that

$$\left.\begin{aligned} (\boldsymbol{\omega} \times \mathbf{v}_r)_x &\cong -\dot{y}\, \omega \sin \lambda \\ (\boldsymbol{\omega} \times \mathbf{v}_r)_y &\cong \dot{x}\, \omega \sin \lambda \\ (\boldsymbol{\omega} \times \mathbf{v}_r)_z &\cong -\dot{y}\, \omega \cos \lambda \end{aligned}\right\} \qquad \textbf{(10.44)}$$

Thus, the equations of interest are

$$\left.\begin{aligned}(\mathbf{a}_r)_x &= \ddot{x} \cong -\frac{T}{m}\cdot\frac{x}{l} + 2\dot{y}\omega\sin\lambda \\ (\mathbf{a}_r)_y &\cong \ddot{y} \cong -\frac{T}{m}\cdot\frac{y}{l} - 2\dot{x}\omega\sin\lambda\end{aligned}\right\} \tag{10.45}$$

For small displacements, $T \cong mg$. Defining $\alpha^2 \equiv T/ml \cong g/l$, and writing $\omega_z = \omega \sin \lambda$, we have

$$\left.\begin{aligned}\ddot{x} + \alpha^2 x &\cong 2\omega_z\dot{y} \\ \ddot{y} + \alpha^2 y &\cong -2\omega_z\dot{x}\end{aligned}\right\} \tag{10.46}$$

We note that the equation for $\ddot{x}$ contains a term in $\dot{y}$ and that the equation for $\ddot{y}$ contains a term in $\dot{x}$. Such equations are called **coupled equations.** A solution for this pair of coupled equations can be effected by adding the first of the above equations to i times the second:

$$(\ddot{x} + i\ddot{y}) + \alpha^2(x + iy) \cong -2\omega_z(i\dot{x} - \dot{y}) = -2i\omega_z(\dot{x} + i\dot{y})$$

If we write

$$q \cong x + iy$$

we then have

$$\ddot{q} + 2i\omega_z\dot{q} + \alpha^2 q \cong 0$$

This equation is identical with the equation that describes damped oscillations (Equation 3.35), except that here the term corresponding to the damping factor is purely imaginary.

The solution is

$$q(t) \cong \exp[-i\omega_z t]\left[A \exp\left(\sqrt{-\omega_z^2 - \alpha^2}\, t\right) + B \exp\left(-\sqrt{-\omega_z^2 - \alpha^2}\, t\right)\right] \quad \textbf{(10.47)}$$

If Earth were not rotating, so that $\omega_z = 0$, then the equation for q would become

$$\ddot{q}' + \alpha^2 q' \cong 0, \qquad \omega_z = 0$$

from which it is seen that α corresponds to the oscillation frequency of the pendulum. This frequency is clearly much greater than the angular frequency of Earth's rotation. Therefore, $\alpha \gg \omega_z$, and the equation for $q(t)$ becomes

$$q(t) \cong e^{-i\omega_z t}(Ae^{i\alpha t} + Be^{-i\alpha t}) \quad \textbf{(10.48)}$$

We can interpret this equation more easily if we note that the equation for q' has the solution

$$q'(t) = x'(t) + iy'(t) = Ae^{i\alpha t} + Be^{-i\alpha t}$$

Thus,

$$q(t) = q'(t) \cdot e^{-i\omega_z t}$$

or

$$x(t) + iy(t) = [(x'(t) + iy'(t)] \cdot e^{-i\omega_z t}$$

$$= (x' + iy')(\cos \omega_z t - i \sin \omega_z t)$$

$$= (x' \cos \omega_z t + y' \sin \omega_z t) + i(-x' \sin \omega_z t + y' \cos \omega_z t)$$

Equating real and imaginary parts,

$$\left.\begin{aligned} x(t) &= x' \cos \omega_z t + y' \sin \omega_z t \\ y(t) &= -x' \sin \omega_z t + y' \cos \omega_z t \end{aligned}\right\}$$

We can write these equations in matrix form as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos \omega_z t & \sin \omega_z t \\ -\sin \omega_z t & \cos \omega_z t \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \tag{10.49}$$

from which (x, y) may be obtained from (x', y') by the application of a rotation matrix of the familiar form

$$\boldsymbol{\lambda} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{10.50}$$

Thus, the angle of rotation is $\theta = \omega_z t$, and the plane of oscillation of the pendulum therefore rotates with a frequency $\omega_z = \omega \sin\lambda$. The observation of this rotation gives a clear demonstration of the rotation of Earth.*