Dynamics of a System of Particles

Ch. 9

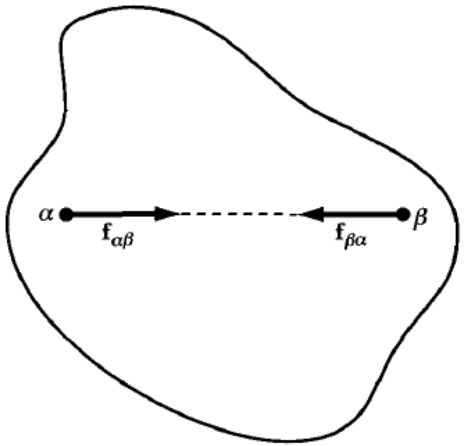
9.1 Introduction

Newton's Third Law plays a prominent role in the dynamics of a system of particles because of the internal forces between the particles in the system. We need to make two assumptions concerning the internal forces:

1. The forces exerted by two particles α and β on each other are equal in magnitude and opposite in direction. Let $\mathbf{f}_{\alpha\beta}$ represent the force on the αth particle due to the βth particle. The so-called "weak" form of Newton's Third Law is

$$\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$$

2. The forces exerted by two particles α and β on each other, in addition to being equal and opposite, must lie on the straight line joining the two particles. This more restrictive form of Newton's Third Law, often called the "strong" form, is displayed in



9.2 Center of Mass

We now extend our discussion from a single particle to a system of n particles. The mass of this system is denoted by M:

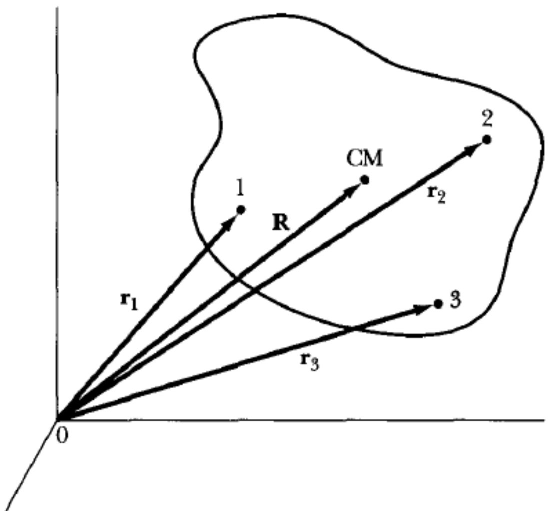
$$M = \sum_{\alpha} m_{\alpha}$$

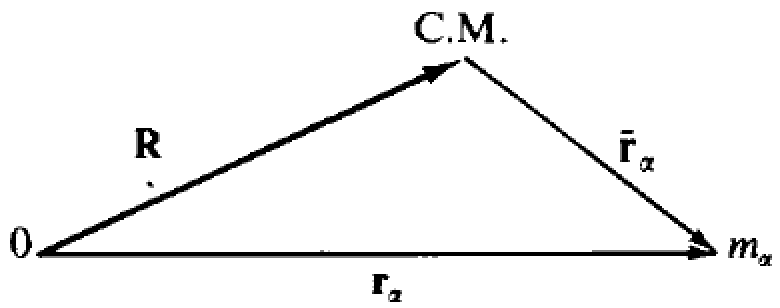
where the summation over a runs from α = 1 to α = *n.* If the vector connecting the origin with the α th particle is r_{α} , then the vector which defines the position of the center of mass of the system is

$$\boxed{\mathbf{R} = \frac{1}{M}\sum_{\alpha} m_{\alpha}\mathbf{r}_{\alpha}}$$

Since it is often convenient to specify the position of a particle with respect to the center of mass *as*

$$\bar{\mathbf{r}}_{\alpha} = \mathbf{r}_{\alpha} - \mathbf{R}$$





For a continuous distribution of mass, the summation is replaced by an integral,

$$\mathbf{R} = \frac{1}{M}\int \mathbf{r}\, dm \tag{9.4}$$

The location of the center of mass of a body is uniquely defined, but the position vector **R** depends on the coordinate system chosen. If the origin in Figure 9-2 were chosen elsewhere, the vector **R** would be different.

EXAMPLE 9.1

Find the center of mass of a solid hemisphere of constant density.

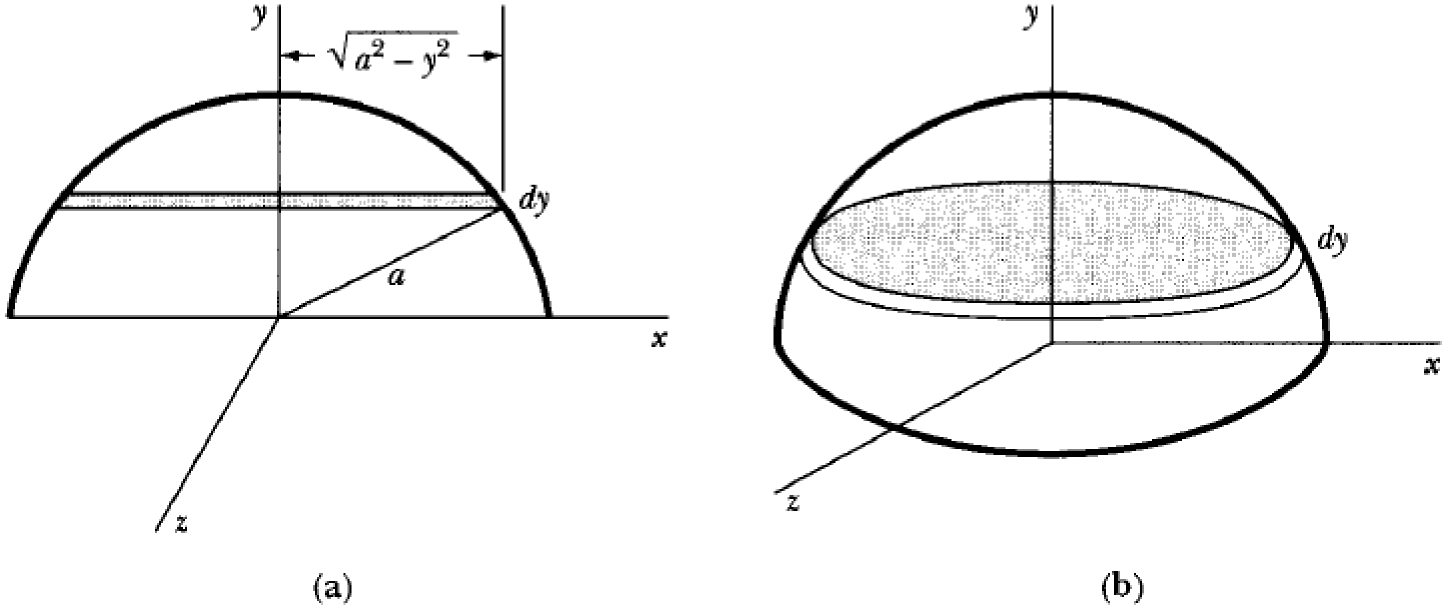
Solution. Let the density be ρ, the hemispherical mass be M, and the radius be a.

$$\rho = \frac{M}{\frac{2}{3}\pi a^3}$$

We want to choose the origin of our coordinate system carefully (Figure 9-3) to make the problem as simple as possible. The position coordinates of R are (X, Y, Z). From symmetry, $X = 0$, $Z = 0$. This should be obvious from Equation 9.4,

$$X = \frac{1}{M}\int_{-a}^{a} x \, dm$$

$$Z = \frac{1}{M}\int_{-a}^{a} z \, dm$$



because we are integrating over an odd power of a variable with symmetric limits. For Y, however, the limits are asymmetric.

$$Y = \frac{1}{M}\int_0^a y\,dm$$

Construct dm so it is placed at a constant value of y. A circular slice perpendicular to the y-axis suffices (see Figure 9-3).

$$dm = \rho dV = \rho\pi(a^2 - y^2)\,dy$$

$$Y = \frac{1}{M}\int_0^a \rho\pi y(a^2 - y^2)\,dy$$

$$Y = \frac{\pi\rho a^4}{4M} = \frac{3a}{8}$$

The position of the center of mass is $(0, 3a/8, 0)$.

9.3 Linear Momentum of a System

One part is the resultant of all forces whose origin lies outside of the system; this is called the **external force, $\mathbf{F}_\alpha^{(e)}$**. The other part is the resultant of the forces arising from the interaction of all of the other $n - 1$ particles with the αth particle; this is called the **internal force, $\mathbf{f}_\alpha$**. Force $\mathbf{f}_\alpha$ is given by the vector sum of all the individual forces $\mathbf{f}_{\alpha\beta}$,

$$\mathbf{f}_\alpha = \sum_\beta \mathbf{f}_{\alpha\beta} \tag{9.5}$$

where $\mathbf{f}_{\alpha\beta}$ represents the force on the αth particle due to the βth particle. The total force acting on the αth particle is therefore

$$\mathbf{F}_\alpha = \mathbf{F}_\alpha^{(e)} + \mathbf{f}_\alpha \tag{9.6}$$

Also, according to the weak statement of Newton's Third Law, we have

$$\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha} \tag{9.1}$$

Newton's Second Law for the αth particle can be written as

$$\dot{\mathbf{p}}_\alpha = m_\alpha \ddot{\mathbf{r}}_\alpha = \mathbf{F}_\alpha^{(e)} + \mathbf{f}_\alpha \tag{9.7}$$

or

$$\frac{d^2}{dt^2}(m_\alpha \mathbf{r}_\alpha) = \mathbf{F}_\alpha^{(e)} + \sum_\beta \mathbf{f}_{\alpha\beta} \tag{9.8}$$

Summing this expression over α, we have

$$\frac{d^2}{dt^2}\sum_\alpha m_\alpha \mathbf{r}_\alpha = \sum_\alpha \mathbf{F}_\alpha^{(e)} + \sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_\beta \mathbf{f}_{\alpha\beta} \tag{9.9}$$

where the terms $\alpha = \beta$ do not enter in the second sum on the right-hand side, because $\mathbf{f}_{\alpha\alpha} \equiv 0$. The summation on the left-hand side just yields $M\mathbf{R}$ (see Equation 9.3), and the second time derivative is $M\ddot{\mathbf{R}}$. The first term on the right-hand side is the sum of all the external forces and can be written as

$$\sum_\alpha \mathbf{F}_\alpha^{(e)} \equiv \mathbf{F} \tag{9.10}$$

The second term on the right-hand side in Equation 9.9 can be expressed* as

$$\sum_{\substack{\alpha \\ \alpha\neq\beta}}\sum_\beta \mathbf{f}_{\alpha\beta} \equiv \sum_{\alpha,\beta\neq\alpha} \mathbf{f}_{\alpha\beta} = \sum_{\alpha<\beta}(\mathbf{f}_{\alpha\beta} + \mathbf{f}_{\beta\alpha})$$

which vanishes† according to Equation 9.1. Thus, we have the first important result

$$M\ddot{\mathbf{R}} = \mathbf{F} \tag{9.11}$$

I. *The center of mass of a system moves as if it were a single particle of mass equal to the total mass of the system, acted on by the total external force, and independent of the nature of the internal forces (as long as they follow $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$, the weak form of Newton's Third Law).*

The total linear momentum of the system is

$$\mathbf{P} = \sum_{\alpha} m_\alpha \dot{\mathbf{r}}_\alpha = \frac{d}{dt}\sum_{\alpha} m_\alpha \mathbf{r}_\alpha = \frac{d}{dt}(M\mathbf{R}) = M\dot{\mathbf{R}} \tag{9.12}$$

and

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F} \tag{9.13}$$

Thus, the total linear momentum of the system is conserved if there is no external force. From Equations 9.12 and 9.13, we note our second and third important results:

II. *The linear momentum of the system is the same as if a single particle of mass* M *were located at the position of the center of mass and moving in the manner the center of mass moves.*

III. *The total linear momentum for a system free of external forces is constant and equal to the linear momentum of the center of mass* (the law of conservation of linear momentum for a system).

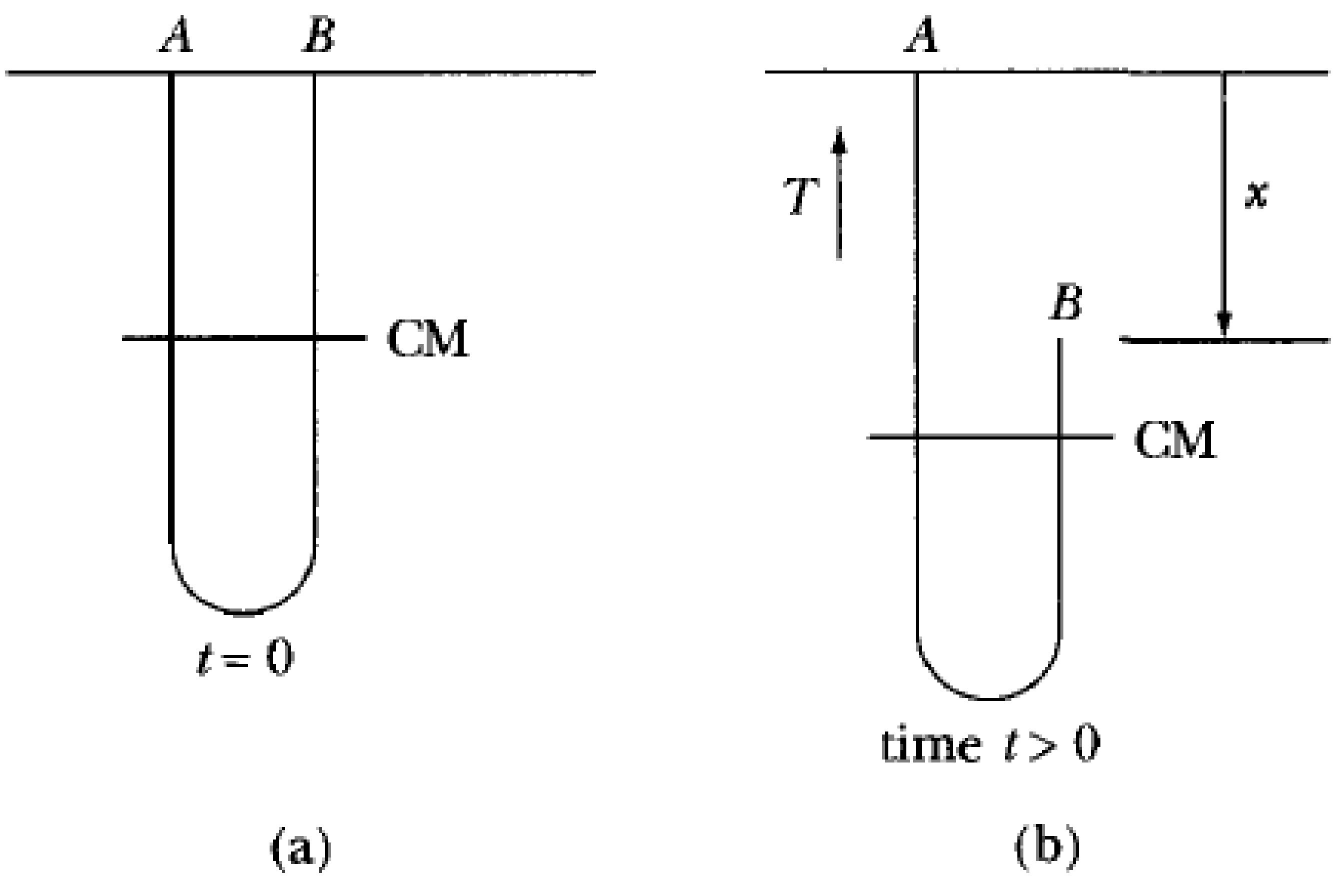
EXAMPLE 9.2

A chain of uniform linear mass density ρ, length b, and mass M ($\rho = M/b$) hangs as shown in Figure 9-4. At time $t = 0$, the ends A and B are adjacent, but end B is released. Find the tension in the chain at point A after end B has fallen a distance x by (a) assuming free fall and (b) by using energy conservation.

Solution. (a) In the case of free fall, let's assume the only forces acting on the system at time t are the tension T acting vertically upward at point A and the gravitational force Mg pulling the chain down. The center of mass momentum reacts to these forces such that

$$\dot{P} = Mg - T \tag{9.14}$$

The right side of the chain, with mass $\rho(b - x)/2$, is moving at the speed $\dot{x}$, and the left side of the chain is not moving. The total momentum of the system is



Example 9.2. (a) A chain of uniform linear mass density hangs at points A and B before B is released at time $t = 0$. (b) At time t the end B has fallen a distance x.

therefore

$$P = \rho\left(\frac{b - x}{2}\right)\dot{x}$$

and

$$\dot{P} = \frac{\rho}{2}[-\dot{x}^2 + \ddot{x}(b - x)] \qquad \mathbf{(9.15)}$$

For free fall, we have $x = gt^2/2$, so that

$$\dot{x} = gt = \sqrt{2gx}$$

$$\ddot{x} = g$$

and

$$\dot{P} = \frac{\rho}{2}(gb - 3gx) = Mg - T$$

and finally,

$$T = \frac{Mg}{2}\left(\frac{3x}{b} + 1\right) \qquad \mathbf{(9.16)}$$

9.4 Angular Momentum of the System

The angular momentum of the αth particle about the origin is given by

$$\mathbf{L}_\alpha = \mathbf{r}_\alpha \times \mathbf{p}_\alpha$$

Summing this expression over α,

$$\mathbf{L} = \sum_\alpha \mathbf{L}_\alpha = \sum_\alpha (\mathbf{r}_\alpha \times \mathbf{p}_\alpha) = \sum_\alpha (\mathbf{r}_\alpha \times m_\alpha \dot{\mathbf{r}}_\alpha)$$

$$= \sum_\alpha (\bar{\mathbf{r}}_\alpha + \mathbf{R}) \times m_\alpha (\dot{\bar{\mathbf{r}}}_\alpha + \dot{\mathbf{R}})$$

$$= \sum_\alpha m_\alpha [(\bar{\mathbf{r}}_\alpha \times \dot{\bar{\mathbf{r}}}_\alpha) + (\bar{\mathbf{r}}_\alpha \times \dot{\mathbf{R}}) + (\mathbf{R} \times \dot{\bar{\mathbf{r}}}_\alpha) + (\mathbf{R} \times \dot{\mathbf{R}})]$$

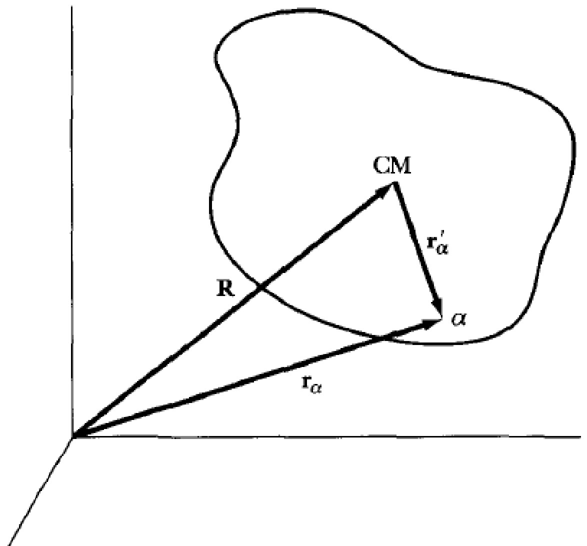
The middle two terms can be written as

$$\left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha\right) \times \dot{\mathbf{R}} + \mathbf{R} \times \frac{d}{dt}\left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha\right)$$

The middle two terms vanishes since

$$\sum_\alpha m_\alpha \bar{\mathbf{r}}_\alpha = \sum_\alpha m_\alpha (\mathbf{r}_\alpha - \mathbf{R}) = \sum_\alpha m_\alpha \mathbf{r}_\alpha - \mathbf{R} \sum_\alpha m_\alpha$$

$$= M\mathbf{R} - M\mathbf{R} \equiv 0$$



That is, $\Sigma_\alpha\, m_\alpha\ r'_\alpha$ specifies the position of the center of mass in the center-of mass coordinate system, and is therefore a null vector. Thus,

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_\alpha \bar{\mathbf{r}}_\alpha \times \bar{\mathbf{p}}_\alpha$$

$$= \mathbf{R} \times \mathbf{P} + \sum_\alpha \bar{\mathbf{r}}_\alpha \times \bar{\mathbf{p}}_\alpha$$

and the total angular momentum is the sum of the angular momentum of the center of mass about the origin and the angular momentum of the system about the position of the center of mass.

The time derivative of the angular momentum of the ath particle is,

$$\dot{\mathbf{L}}_\alpha = \mathbf{r}_\alpha \times \dot{\mathbf{p}}_\alpha$$

$$\dot{\mathbf{L}}_\alpha = \mathbf{r}_\alpha \times (\mathbf{F}_\alpha^{(e)} + \sum_\beta \mathbf{f}_{\alpha\beta})$$

Summing this expression over a, we have

$$\dot{\mathbf{L}} = \sum_\alpha \dot{\mathbf{L}}_\alpha = \sum_\alpha (\mathbf{r}_\alpha \times \mathbf{F}_\alpha^{(e)}) + \sum_{\alpha,\, \beta \neq \alpha} (\mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta})$$

It is easy to verify that the last term may be written as

$$\sum_{\alpha, \beta \neq \alpha} (\mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta}) = \sum_{\alpha < \beta} [(\mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta}) + (\mathbf{r}_\beta \times \mathbf{f}_{\beta\alpha})]$$

Now, the vector connecting the αth and βth particles is defined to be

$$\mathbf{r}_{\alpha\beta} \equiv \mathbf{r}_\alpha - \mathbf{r}_\beta$$

and then since $\mathbf{f}_{\alpha\beta} = -\mathbf{f}_{\beta\alpha}$, we have

$$\sum_{\alpha, \beta \neq \alpha} (\mathbf{r}_\alpha \times \mathbf{f}_{\alpha\beta}) = \sum_{\alpha < \beta} (\mathbf{r}_\alpha - \mathbf{r}_\beta) \times \mathbf{f}_{\alpha\beta}$$

$$= \sum_{\alpha < \beta} (\mathbf{r}_{\alpha\beta} \times \mathbf{f}_{\alpha\beta})$$

But, since we have limited the discussion to the case of central forces, $\mathbf{f}_{\alpha\beta}$ is directed along the line joining m_α with m_β, i.e., along $\mathbf{r}_{\alpha\beta}$. Hence,

$$\mathbf{r}_{\alpha\beta} \times \mathbf{f}_{\alpha\beta} \equiv 0$$

and

$$\dot{\mathbf{L}} = \sum_\alpha (\mathbf{r}_\alpha \times \mathbf{F}_\alpha^{(e)})$$

The right-hand side of this expression is just the sum of all of the external torques:

$$\dot{\mathbf{L}} = \sum_\alpha \mathbf{N}_\alpha^{(e)} = \mathbf{N}^{(e)}$$

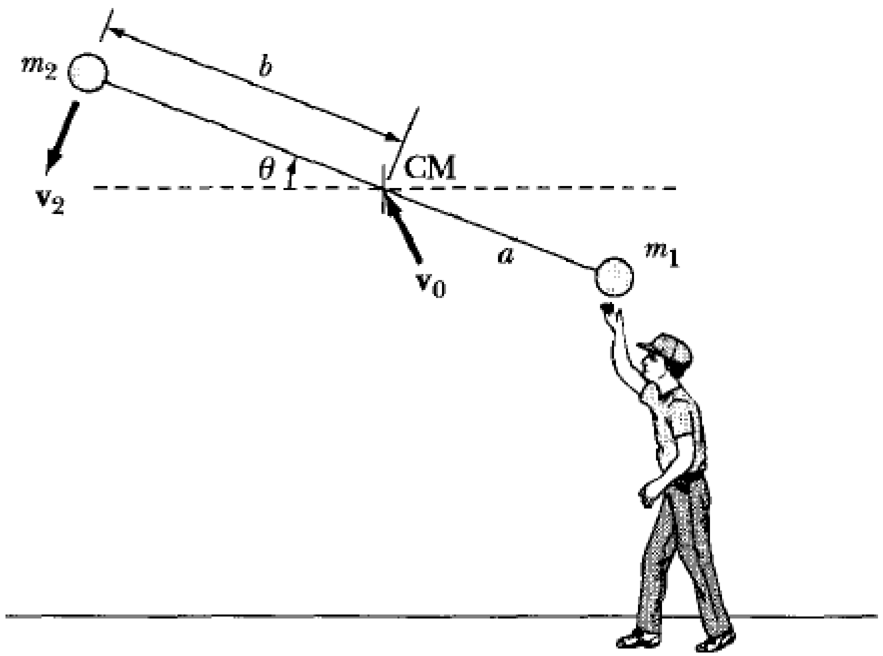
Thus, *if the external torques about a given axis vanish, then the total angular momentum of the system about that axis remains constant in time.*

we may then state that *the total internal torque must vanish if the internal forces are central in character, i.e., if* $f_{\alpha\beta} = - f_{\beta\alpha}$, *and the angular momentum of an isolated system cannot be altered without the application of external forces.*

EXAMPLE 9.3

A light string of length a has bobs of mass m_1 and $m_2 (m_2 > m_1)$ on its ends. The end with m_1 is held and m_2 is whirled vigorously by hand above the head in a counterclockwise direction (looking down from above) and then released.

Describe the subsequent motion, and find the tension in the string after release.



Solution. The system is shown in Figure 9-7. The center of mass is a distance $b = [m_1/(m_1 + m_2)]a$ from mass m_2. After being released, the only forces on the system are the gravitational forces on m_1 and m_2. Assume that $\mathbf{v}_0$ is the initial velocity of the center of mass CM. The CM will continue in a parabolic path under the influence of gravity as if all the mass $(m_1 + m_2)$ were concentrated at the CM. But when released, mass m_2 is rotating around m_1 rapidly. Because no external torque exists, the system will continue to rotate. But now both m_1 and m_2 rotate about the CM, and the angular momentum is conserved. If mass m_2 is traveling with the linear velocity $\mathbf{v}_2$ when released, then we must have $v_2 = b\dot{\theta}$ [similarly, $v_1 = (a - b)\dot{\theta}$]. The tension in the string is, however, due to the centrifugal reaction of the masses rotating, which is, in this case,

$$\text{Centrifugal force} = \frac{m_2(b\dot{\theta})^2}{b} = \text{Tension}$$

$$\text{Tension} = m_2 b\dot{\theta}^2 = m_2\left(\frac{m_1 a}{m_1 + m_2}\right)\dot{\theta}^2 = \frac{m_1 m_2 a\dot{\theta}^2}{m_1 + m_2}$$

9.5 Energy of the System

The final conservation theorem, that of energy, may be derived for a system of particles as follows .

Consider the work done on the system in moving it from a Configuration 1, in which all the coordinates $\mathbf{r}_\alpha$ are specified, to a Configuration 2, in which the coordinates $\mathbf{r}_\alpha$ have some different specification.

We can write the total work done on the system as the sum of the work done on individual particles

$$W_{12} = \sum_{\alpha} \int_1^2 \mathbf{F}_\alpha \bullet d\mathbf{r}_\alpha$$

$$
\begin{aligned}
W_{12} &= \sum_{\alpha} \int_1^2 \mathbf{F}_\alpha \bullet d\mathbf{r}_\alpha \\
&= \sum_{\alpha} \int_1^2 m_\alpha \frac{d\mathbf{v}_\alpha}{dt} \bullet \frac{d\mathbf{r}_\alpha}{dt} dt = \sum_{\alpha} \int_1^2 m_\alpha \frac{d\mathbf{v}_\alpha}{dt} \bullet \mathbf{v}_\alpha dt \\
&= \sum_{\alpha} \int_1^2 \frac{1}{2} m_\alpha \frac{dv_\alpha{}^2}{dt} dt = \sum_{\alpha} \int_1^2 \frac{d}{dt}\left(\frac{1}{2} m_\alpha v_\alpha{}^2 \right) dt \\
&= \sum_{\alpha} \int_1^2 d\left(\frac{1}{2} m_\alpha v_\alpha{}^2 \right) = T_2 - T_1,
\end{aligned}
$$

where $$T = \sum_{\alpha} T_\alpha = \sum_{\alpha} \frac{1}{2} m_\alpha v_\alpha^2$$

Using the relation $\dot{\mathbf{r}}_\alpha = \dot{\mathbf{r}}'_\alpha + \dot{\mathbf{R}}$

we have

$$\dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha = v_\alpha^2 = (\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{R}}) \cdot (\dot{\mathbf{r}}'_\alpha + \dot{\mathbf{R}})$$

$$= (\dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{r}}'_\alpha) + 2(\dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{R}}) + (\dot{\mathbf{R}} \cdot \dot{\mathbf{R}})$$

$$= v_\alpha'^2 + 2(\dot{\mathbf{r}}'_\alpha \cdot \dot{\mathbf{R}}) + V^2$$

where $\mathbf{v}' \equiv \dot{\mathbf{r}}'$ and where V is the velocity of the center of mass. Then

$$T = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 \quad = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha'^2 + \sum_\alpha \frac{1}{2} m_\alpha V^2 + \dot{\mathbf{R}} \cdot \frac{d}{dt} \sum_\alpha m_\alpha \mathbf{r}'_\alpha$$

But, by a previous argument, $\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0$, and the last term vanishes. Thus,

$$\boxed{T = \sum \frac{1}{2} m_\alpha v_\alpha'^2 + \frac{1}{2} M V^2} \quad \rightarrow \quad (1)$$

which can be stated:

VII. *The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with the velocity of the center of mass and the kinetic energy of motion of the individual particles relative to the center of mass.*

In another way , the work done can be written as follow :

$$W_{12} = \sum_{\alpha} \int_1^2 \mathbf{F}_\alpha^{(e)} \cdot d\mathbf{r}_\alpha + \sum_{\alpha, \beta \neq \alpha} \int_1^2 \mathbf{f}_{\alpha\beta} \cdot d\mathbf{r}_\alpha$$

If the forces $\mathbf{F}_\alpha^{(e)}$ and $\mathbf{f}_{\alpha\beta}$ are conservative, then they are derivable from potential functions, and we can write

The first term in equation of work becomes

$$\sum_{\alpha}\int_{1}^{2}\mathbf{F}_{\alpha}^{(e)}\cdot d\mathbf{r}_{\alpha} = -\sum_{\alpha}\int_{1}^{2}(\nabla_{\alpha}U_{\alpha})\cdot d\mathbf{r}_{\alpha}$$

$$= -\sum_{\alpha}U_{\alpha}\Big|_{1}^{2}$$

And the second term in equation of work becomes

$$\sum_{\alpha,\beta\neq\alpha}\int_{1}^{2}\mathbf{f}_{\alpha\beta}\cdot d\mathbf{r}_{\alpha} = -\sum_{\alpha<\beta}\int_{1}^{2}d\overline{U}_{\alpha\beta} = -\sum_{\alpha<\beta}\overline{U}_{\alpha\beta}\Big|_{1}^{2}$$

The total work becomes

$$\boxed{W_{12} = -\sum_{\alpha}U_{\alpha}\Big|_{1}^{2} - \sum_{\alpha<\beta}\overline{U}_{\alpha\beta}\Big|_{1}^{2}} \quad \rightarrow \quad (2)$$

We obtained this equation assuming that both the external and internal forces were derivable from potentials. In such a case, the *total potential energy* (both internal and external) for the system can be written as

$$U = \sum_{\alpha} U_{\alpha} + \sum_{\alpha<\beta} \overline{U}_{\alpha\beta}$$

Then,

$$W_{12} = -U\Big|_1^2 = U_1 - U_2$$

From equation (1) and (2)

$$T_2 - T_1 = U_1 - U_2$$

or

$$T_1 + U_1 = T_2 + U_2$$

$$\boxed{E_1 = E_2}$$

which expresses the conservation of energy for the system. This result is valid for a system in which all the forces are derivable from potentials that do not depend explicitly on the time; we say that such a system is *conservative.*

VIII. *The total energy for a conservative system is constant.*

EXAMPLE 9.4

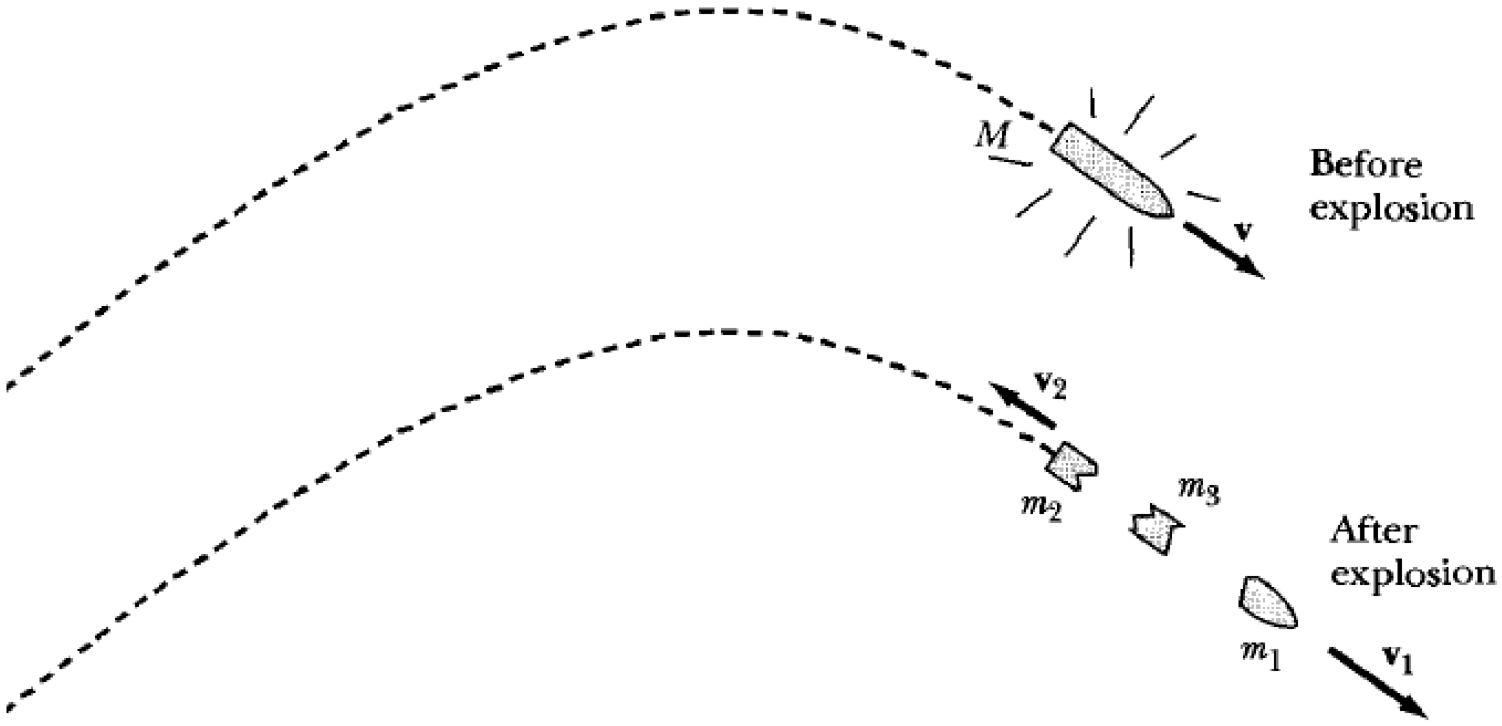
A projectile of mass M explodes while in flight into three fragments One mass ($m_1 = M/2$) travels in the original direction of the projectile, mass m_2 ($= M/6$) travels in the opposite direction, and mass m_3 ($= M/3$) comes to rest. The energy E released in the explosion is equal to five times the projectile's kinetic energy at explosion. What are the velocities?

Solution. Let the velocity of the projectile of mass M be $\mathbf{v}$. The three fragments have the following masses and velocities:

$$m_1 = \frac{M}{2}, \quad \mathbf{v}_1 = k_1\mathbf{v} \qquad \text{Forward direction, } k_1 > 0$$

$$m_2 = \frac{M}{6}, \quad \mathbf{v}_2 = -k_2\mathbf{v} \qquad \text{Opposite direction, } k_2 > 0$$

$$m_3 = \frac{M}{3}, \quad \mathbf{v}_3 = 0 \qquad \text{At rest}$$



Example 9.4. A projectile of mass M explodes in flight into three fragments of masses m_1, m_2, and m_3.

The conservation of linear momentum and energy give

$$Mv = \frac{M}{2} k_1 v - \frac{M}{6} k_2 v \tag{1}$$

$$E + \frac{1}{2} M v^2 = \frac{1}{2} \frac{M}{2} (k_1 v)^2 + \frac{1}{2} \frac{M}{6} (k_2 v)^2 \tag{2}$$

From Equation (1), $k_2 = 3k_1 - 6$, which we can insert into Equation (2):

$$5\left(\frac{1}{2} M v^2\right) + \frac{1}{2} M v^2 = \frac{Mv^2}{4} k_1^2 + \frac{Mv^2}{12} (3k_1 - 6)^2$$

which reduces to $k_1^2 - 3k_1 = 0$, giving the results $k_1 = 0$ and $k_1 = 3$. For $k_1 = 0$, the value of $k_2 = -6$, which is inconsistent with $k_2 > 0$. For $k_1 = 3$, the value of $k_2 = 3$. The velocities become

$$\mathbf{v}_1 = 3\mathbf{v}$$

$$\mathbf{v}_2 = -3\mathbf{v}$$

$$\mathbf{v}_3 = 0$$