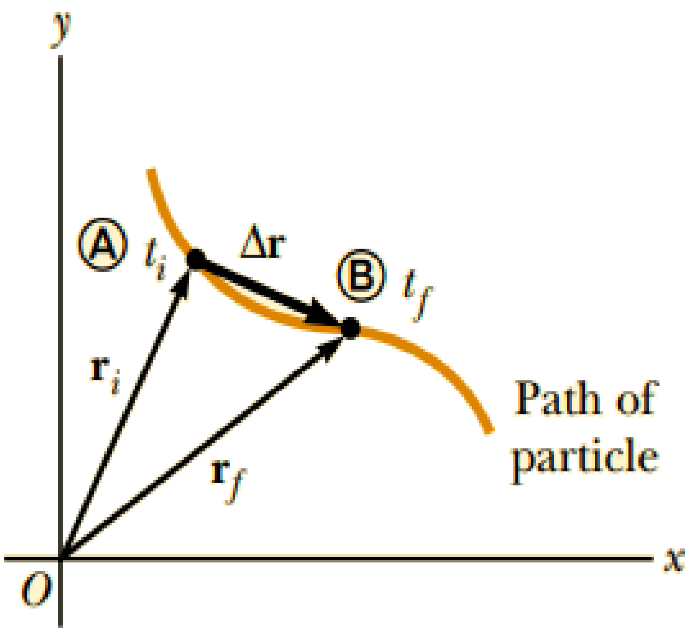
Motion in Two Dimensions

Chapter 4

3.1 The Position, Velocity, and Acceleration Vectors

- In two- or three-dimensional kinematics, everything is the same as in one-dimensional motion except that we must now use full vector notation
- **The displacement** vector Δr for a particle is defined as the difference between its final position vector and its initial position vector



$$\Delta \mathbf{r} \equiv \mathbf{r}_f - \mathbf{r}_i$$

Average Velocity

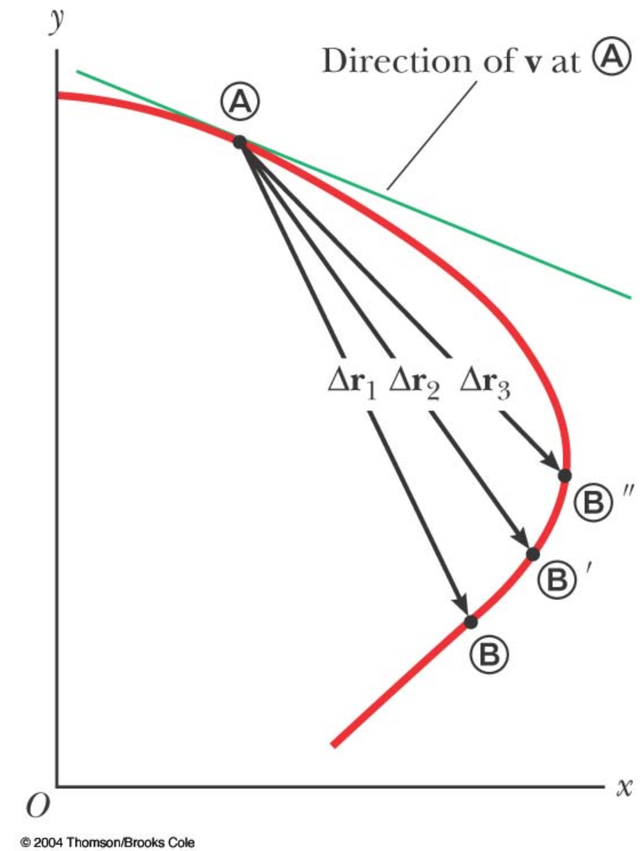
- The average velocity is the ratio of the displacement to the time interval for the displacement

$$\overline{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t}$$

  - The direction of the average velocity is the direction of the displacement vector, $\Delta \mathbf{r}$

Instantaneous Velocity

- The instantaneous velocity is the limit of the average velocity as Δt approaches zero
- The direction of the instantaneous velocity is along a line that is tangent to the path of the particle's direction of motion
- The magnitude of the instantaneous velocity vector is the speed



$$\mathbf{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

Acceleration

- **The average acceleration** of a particle as it moves is defined as the change in the instantaneous velocity vector divided by the time interval during which that change occurs.

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_\mathrm{f} - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

- **The instantaneous acceleration** is the limit of the average acceleration as $\Delta \mathbf{v}/\Delta t$ approaches zero

$$\mathbf{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

Producing An Acceleration

- Various changes in a particle's motion may produce an acceleration
  - The magnitude of the velocity vector may change
  - The direction of the velocity vector may change
    - Even if the magnitude remains constant
  - Both may change simultaneously

4.2 Two-Dimensional Motion with Constant Acceleration

- When the two-dimensional motion has a constant acceleration, the equations of motion will be similar to those of one-dimensional kinematics
  - $\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$ becomes

    $v_{xf} = v_{xi} + a_x t$ and

    $v_{yf} = v_{yi} + a_y t$
  - $\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + ½\,\mathbf{a}t^2$ becomes

    $x_f = x_i + v_{xi} t + ½\,a_x t^2$ and

    $y_f = y_i + v_{yi} t + ½\,a_y t^2$

4.3 Projectile Motion

- An object may move in both the x and y directions simultaneously
- The form of two-dimensional motion we will deal with is called **projectile motion**

Assumptions of Projectile Motion

- The free-fall acceleration **g** is constant over the range of motion
  - And is directed downward
- The effect of air friction is negligible
- With these assumptions, an object in projectile motion will follow a parabolic path
  - This path is called the ***trajectory***

Verifying the Parabolic Trajectory

- Reference frame chosen
  - *y* is vertical with upward positive
- Acceleration components
  - $a_y = -g$ and $a_x = 0$
- Initial velocity components
  - $v_{xi} = v_i \cos\theta$ and $v_{yi} = v_i \sin\theta$

Verifying the Parabolic Trajectory, cont

- Displacements
  - $x_f = v_{xi}\, t = (v_i \cos\,\theta)\; t$
  - $y_f = v_{yi}\, t + ½ a_y\, t^2 = (v_i \sin\,\theta) t - ½\, gt^2$
- Combining the equations gives:

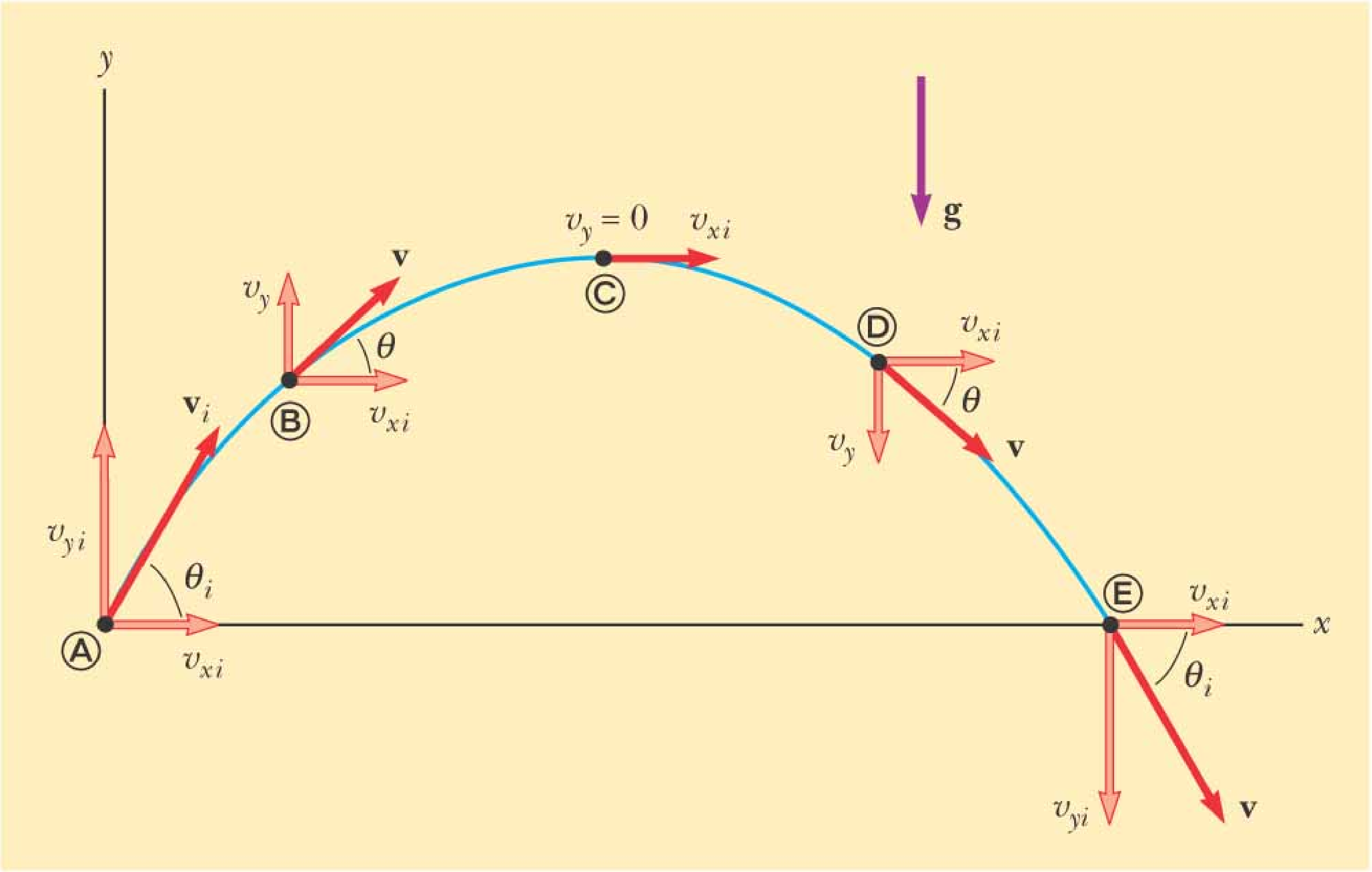
$$y = \left(\tan\theta_i\right)x - \left(\frac{g}{2v_i^2\cos^2\theta_i}\right)x^2$$

  - This is in the form of $y = ax - bx^2$ which is the standard form of a parabola

Analyzing Projectile Motion

- Consider the motion as the superposition of the motions in the *x*- and *y*-directions
- The *x*-direction has constant velocity
  - $a_x = 0$
- The y-direction is free fall
  - $a_y = -g$

Projectile Motion Diagram

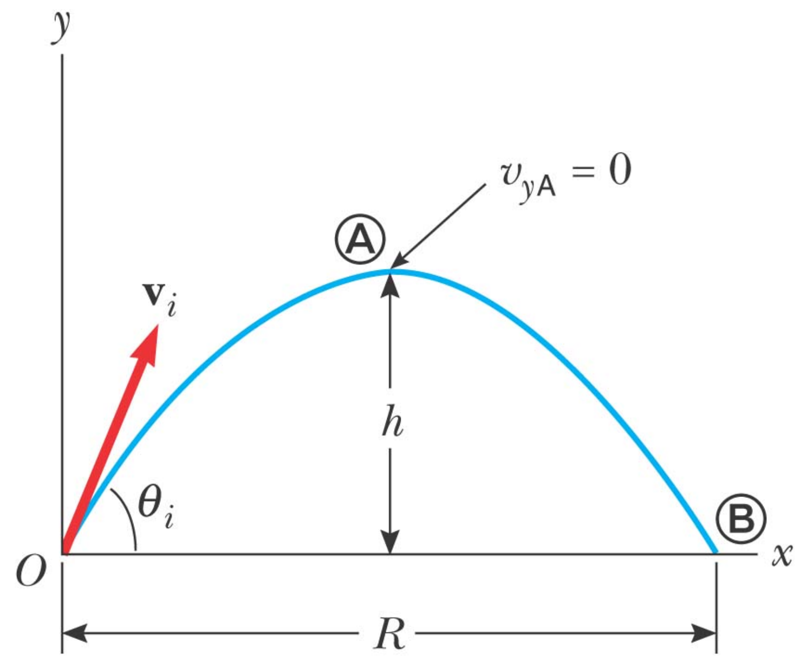


Projectile Motion – Implications

- The *y*-component of the velocity is zero at the maximum height of the trajectory
- The accleration stays the same throughout the trajectory

Range and Maximum Height of a Projectile

- When analyzing projectile motion, two characteristics are of special interest
- The range, *R*, is the horizontal distance of the projectile
- The maximum height the projectile reaches is *h*



Height of a Projectile, equation

- The maximum height of the projectile can be found in terms of the initial velocity vector:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

- This equation is valid only for symmetric motion

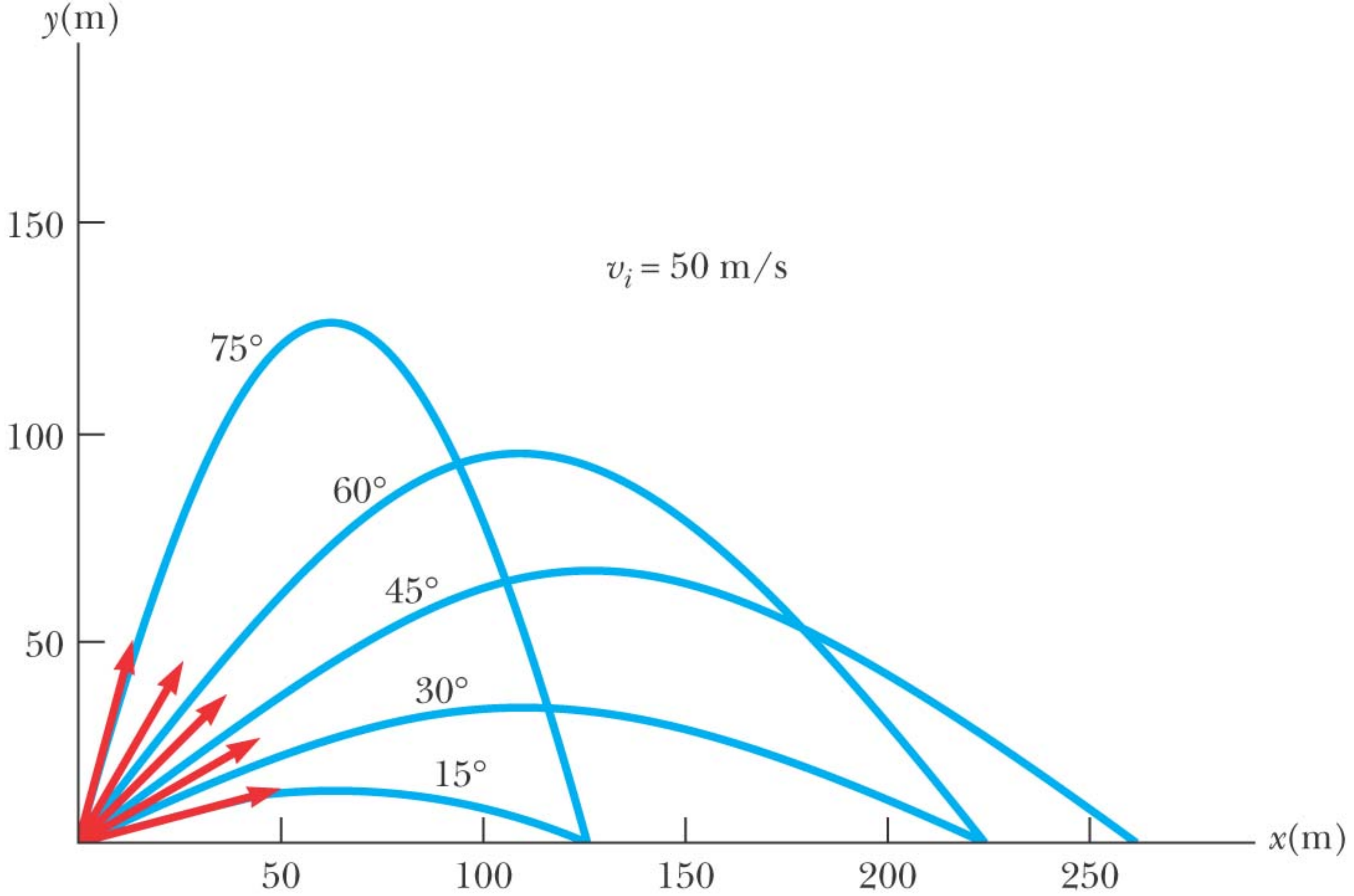
Range of a Projectile, equation

- The range of a projectile can be expressed in terms of the initial velocity vector:

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

- This is valid only for symmetric trajectory

More About the Range of a Projectile



Range of a Projectile, final

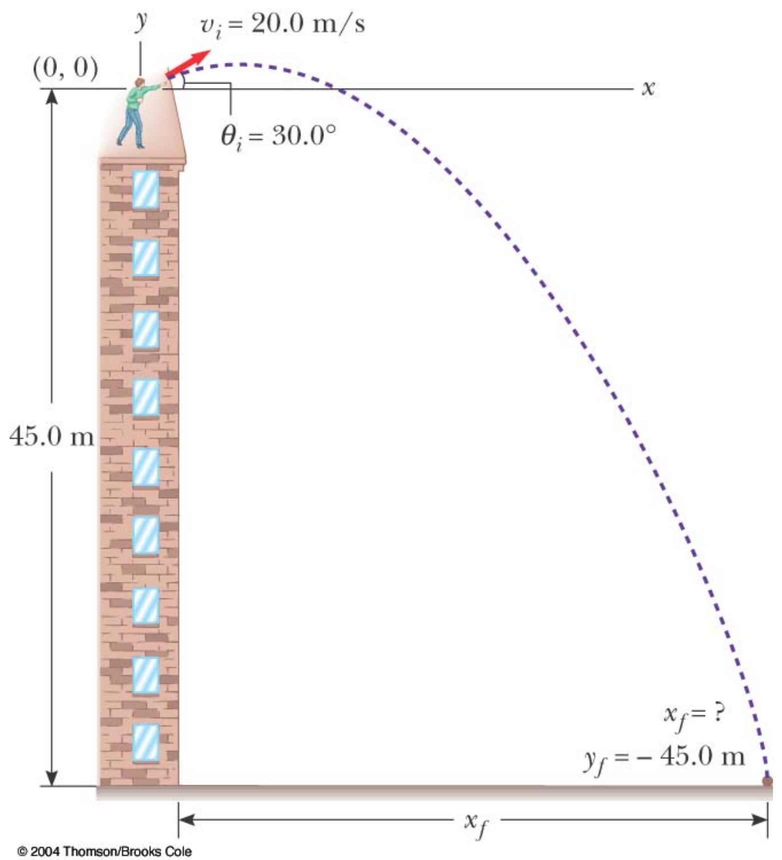
- The maximum range occurs at $\theta_i = 45^o$
- Complementary angles will produce the same range
  - The maximum height will be different for the two angles
  - The times of the flight will be different for the two angles

Projectile Motion – Problem Solving Hints

- Select a coordinate system
- Resolve the initial velocity into x and y components
- Analyze the horizontal motion using constant velocity techniques
- Analyze the vertical motion using constant acceleration techniques
- Remember that both directions share the same time

Non-Symmetric Projectile Motion

- Follow the general rules for projectile motion
- Break the *y*-direction into parts
  - up and down *or*
  - symmetrical back to initial height and then the rest of the height
- May be non-symmetric in other ways

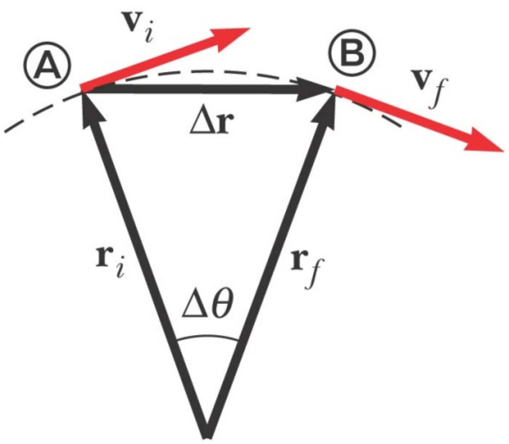


Uniform Circular Motion

- ***Uniform circular motion*** occurs when an object moves in a circular path with a constant speed
- An acceleration exists since the *direction* of the motion is changing
  - This change in velocity is related to an acceleration
- The velocity vector is always tangent to the path of the object

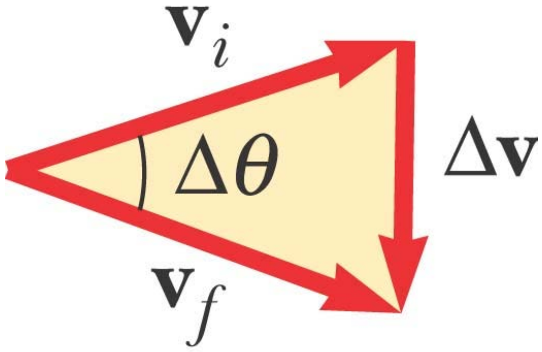
4.4 The Particle in Uniform Circular Motion

- The change in the velocity vector is due to the change in direction



(b)

- The vector diagram shows $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$



Centripetal Acceleration

- The acceleration is always perpendicular to the path of the motion
- The acceleration always points toward the center of the circle of motion
- This acceleration is called the ***centripetal acceleration***

Centripetal Acceleration, cont

- The magnitude of the centripetal acceleration vector is given by

$$a_C = \frac{v^2}{r}$$

- The direction of the centripetal acceleration vector is always changing, to stay directed toward the center of the circle of motion

Period

- The ***period***, *T*, is the time required for one complete revolution
- The speed of the particle would be the circumference of the circle of motion divided by the period
- Therefore, the period is

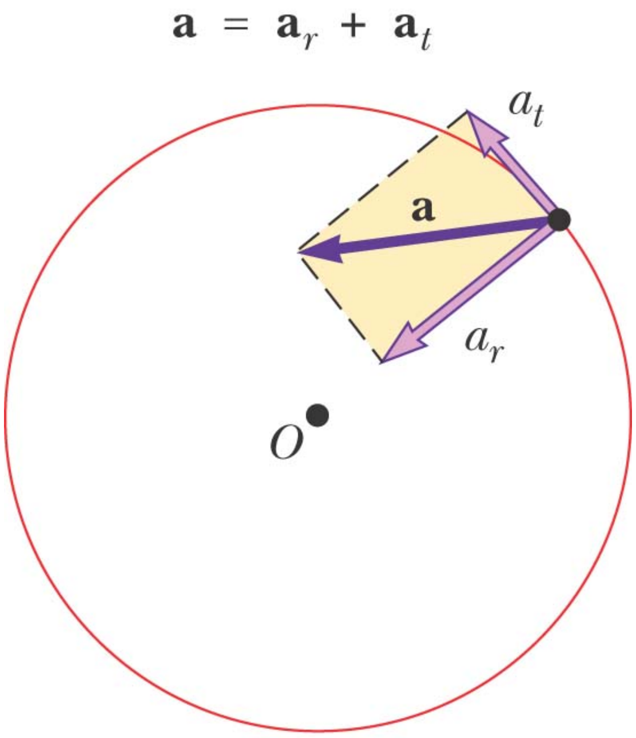
$$T \equiv \frac{2\pi r}{v}$$

Tangential Acceleration

- The magnitude of the velocity could also be changing
- In this case, there would be a ***tangential acceleration***

Total Acceleration

- The tangential acceleration causes the change in the speed of the particle
- The radial acceleration comes from a change in the direction of the velocity vector



(b)

Total Acceleration, equations

- The tangential acceleration:

$$a_t = \frac{d\,|\mathbf{v}|}{dt}$$

- The radial acceleration:

$$a_r = \frac{v^2}{r}$$

- The total acceleration:
  - Magnitude

$$a = \sqrt{a_r^2 + a_t^2}$$

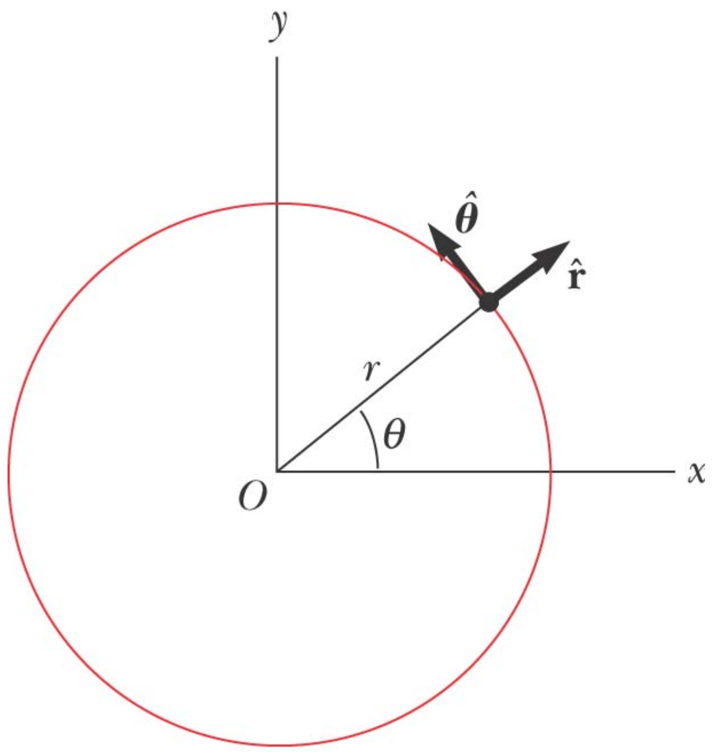
Total Acceleration, In Terms of Unit Vectors

- Define the following unit vectors

$$\hat{\mathbf{r}} \text{ and } \hat{\theta}$$

  - r lies along the radius vector
  - θ is tangent to the circle
- The total acceleration is

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \frac{d\left|\mathbf{v}\right|}{dt}\hat{\theta} - \frac{v^2}{r}\hat{\mathbf{r}}$$



(a)

Problems

- Problem (1) :
- A particle starts from the origin at t =0 with an initial velocity having an x component of 20 m/s and a y component of -15 m/s. The particle moves in the xy plane with an x component of acceleration only, given by ax =4.0 m/s2.
- (A) Determine the total velocity vector at any time.
- (B) Calculate the velocity and speed of the particle at t= 5.0 s.
- (C) Determine the x and y coordinates of the particle at any time t and its position vector at this time.
- Solution:
- (A) To begin the mathematical analysis, we set

$$v_{xi} = 20 \text{ m/s},\ v_{yi} = -15 \text{ m/s},\ a_x = 4.0 \text{ m/s}^2, \text{ and } a_y = 0.$$

$$\vec{\mathbf{v}}_f = \vec{\mathbf{v}}_i + \vec{\mathbf{a}}t = (v_{xi} + a_x t)\hat{\mathbf{i}} + (v_{yi} + a_y t)\hat{\mathbf{j}}$$

$$\vec{\mathbf{v}}_f = [(20 + 4.0t)\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \mathrm{m/s}$$

(B) Evaluate the result at *t=5.0 s:*

$$\vec{\mathbf{v}}_f = [(20 + 4.0(5.0))\hat{\mathbf{i}} - 15\hat{\mathbf{j}}]\ \mathrm{m/s} = (40\hat{\mathbf{i}} - 15\hat{\mathbf{j}})\ \mathrm{m/s}$$

$$\theta = \tan^{-1}\left(\frac{v_{yf}}{v_{xf}}\right) = \tan^{-1}\left(\frac{-15\ \mathrm{m/s}}{40\ \mathrm{m/s}}\right) = -21°$$

- Evaluate the speed of the particle as the magnitude
- of velocity :

$$v_f = |\vec{\mathbf{v}}_f| = \sqrt{v_{xf}^2 + v_{yf}^2} = \sqrt{(40)^2 + (-15)^2}\ \mathrm{m/s} = 43\ \mathrm{m/s}$$

(C)

$$x_f = v_{xi}t + \frac{1}{2}a_x t^2 = (20t + 2.0t^2)\text{ m}$$

$$y_f = v_{yi}t = (-15t)\text{ m}$$

the position vector of the particle at any time *t:*

$$\vec{\mathbf{r}}_f = x_f\hat{\mathbf{i}} + y_f\hat{\mathbf{j}} = [(20t + 2.0t^2)\hat{\mathbf{i}} - 15t\hat{\mathbf{j}}]\text{ m}$$

- Problem (2) :
- A long jumper leaves the ground at an angle of 20.0° above the horizontal and at a speed of 11.0 m/s.
- (A) How far does he jump in the horizontal direction?
- (B) What is the maximum height reached?
- Solution:
- (A) to find the range of the jumper:

$$R = \frac{v_i^2 \sin 2\theta_i}{g} = \frac{(11.0 \text{ m/s})^2 \sin 2(20.0°)}{9.80 \text{ m/s}^2} = 7.94 \text{ m}$$

- (B) the maximum height:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g} = \frac{(11.0 \text{ m/s})^2 (\sin 20.0°)^2}{2(9.80 \text{ m/s}^2)} = 0.722 \text{ m}$$

- Problem (3) :
- A ski jumper leaves the ski track moving in the horizontal direction with a speed of 25.0 m/s. The landing incline below her falls off with a slope of 35.0°. Where does she land on the incline?
- Solution:
- The initial velocity components are

$$v_{xi} = 25.0 \text{ m/s} \quad \text{and} \quad v_{yi} = 0.$$

- From the right triangle, we see that the jumper's x and y coordinates at the landing point are given by

$$x_f = d \cos 35.0° \quad \text{and} \quad y_f = d \sin 35.0°.$$

$$x_f = v_{xi} t = (25.0 \text{ m/s}) t \;\Longrightarrow\; d \cos 35.0° = (25.0 \text{ m/s}) t$$

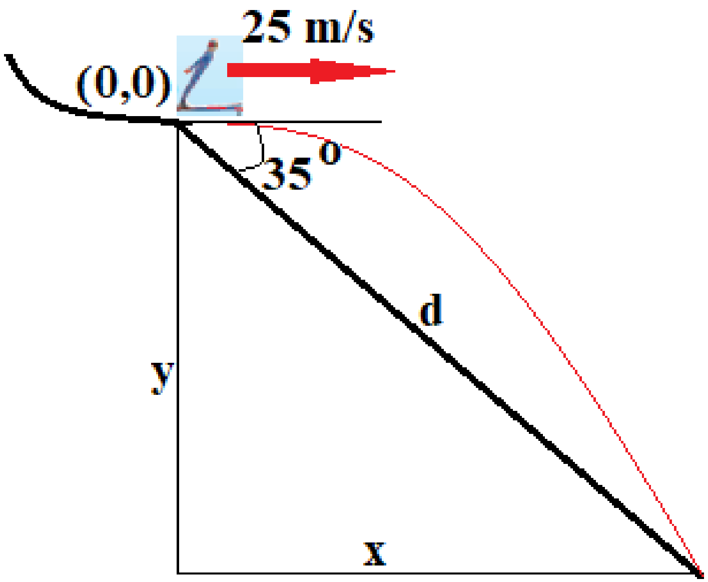
$$\Longrightarrow\; t = \frac{d \cos 35.0°}{25.0 \text{ m/s}}$$

$$y_f = v_{yi}t + \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}(9.80\ \text{m/s}^2)t^2$$

$$-d \sin 35.0° = -\tfrac{1}{2}(9.80\ \text{m/s}^2)t^2$$

$$-d \sin 35.0° = -\tfrac{1}{2}(9.80\ \text{m/s}^2)\left(\frac{d \cos 35.0°}{25.0\ \text{m/s}}\right)^2$$

$$d = 109\ \text{m}$$



- To evaluate the *x and y coordinates of the point at which* the skier lands:

$$x_f = d \cos 35.0° = (109\ \text{m})\cos 35.0° = 89.3\ \text{m}$$

$$y_f = -d \sin 35.0° = -(109\ \text{m})\sin 35.0° = -62.5\ \text{m}$$

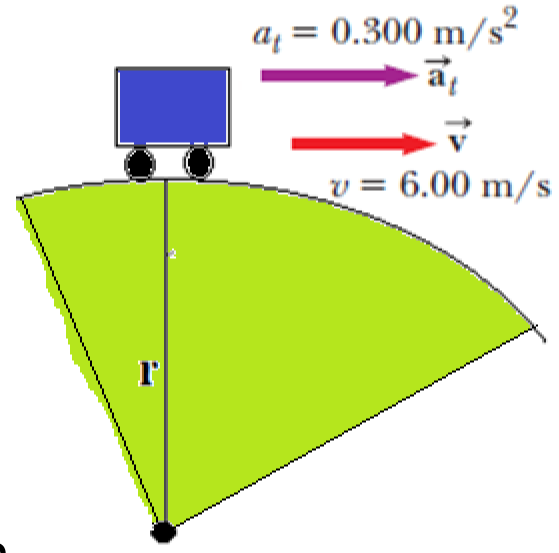
- Problem (4) :
- What is the centripetal acceleration of the Earth as it moves in its orbit around the Sun?
- Solution:
- in terms of the period of the Earth's orbit, which we know is one year, and the radius of the Earth's orbit around the Sun, which is 1.496×10^{11} m.

$$a_c = \frac{v^2}{r} \quad , \quad T = \frac{2\pi r}{v} \quad \Rightarrow \quad a_c = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{T}\right)^2}{r} = \frac{4\pi^2 r}{T^2}$$

$$a_c = \frac{4\pi^2(1.496 \times 10^{11}\ \text{m})}{(1\ \text{yr})^2}\left(\frac{1\ \text{yr}}{3.156 \times 10^7\ \text{s}}\right)^2 = 5.93 \times 10^{-3}\ \text{m/s}^2$$

- Problem (5) :

A car exhibits a constant acceleration of 0.300 m/s2 parallel to the roadway. The car passes over a rise in the roadway such that the top of the rise is shaped like a circle of radius 500 m. At the moment the car is at the top of the rise, its velocity vector is horizontal and has a magnitude of 6.00 m/s. What are the magnitude and direction of the total acceleration vector for the car at this instant?



- Solution:

With v = 6 m/s and r =500 m. The radial acceleration vector is directed to the center of circle, and the tangential acceleration vector has magnitude 0.3 m/s2 and is horizontal.

the radial acceleration:

$$a_r = -\frac{v^2}{r} = -\frac{(6.00\ \mathrm{m/s})^2}{500\ \mathrm{m}} = -0.072\,0\ \mathrm{m/s^2}$$

the magnitude of **a**:

$$\sqrt{a_r^2 + a_t^2} = \sqrt{(-0.072\,0\ \mathrm{m/s^2})^2 + (0.300\ \mathrm{m/s^2})^2}$$
$$= 0.309\ \mathrm{m/s^2}$$

the angle ϕ between **a** and the horizontal:

$$\phi = \tan^{-1}\frac{a_r}{a_t} = \tan^{-1}\left(\frac{-0.072\,0\ \mathrm{m/s^2}}{0.300\ \mathrm{m/s^2}}\right) = -13.5^\circ$$