Central-Force Motion

Ch. 8

8.7 Planetary Motion—Kepler's Problem

The equation for the path of a particle moving under the influence of a central force whose magnitude is inversely proportional to the square of the distance, can be obtained from :

$$\theta(r) = \int \frac{(l/r^2)\, dr}{\sqrt{2\mu\left(E + \frac{k}{r} - \frac{l^2}{2\mu r^2}\right)}} + \text{constant}$$

The integral can be easily evaluated if the variable is changed to *u = l/r*
And if the origin of *Θ is defined so that the integration* constant is zero, we find

Setting $u = 1/r$, Eq. (8.38) can be rewritten as

$$\theta = -\int \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu k}{\ell^2}u - u^2}}$$

Using the standard form of the integral

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{-1}{\sqrt{-a}} \sin^{-1}\left[\frac{2ax + b}{\sqrt{b^2 - 4ac}}\right] + \text{const.}$$

we have

$$\theta + \text{const.} = \sin^{-1}\left[\frac{-\frac{2}{r} + \frac{2\mu k}{\ell^2}}{\sqrt{\left[\frac{2\mu k}{\ell^2}\right]^2 + 8\frac{\mu E}{\ell^2}}}\right]$$

We can choose the point from which θ is measured so that the constant in (4) is $-\pi/2$. Then,

$$\boxed{\cos\theta = \frac{\frac{\ell^2}{\mu k}\frac{1}{r} - 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}}} \quad \Rightarrow \quad \textbf{(1)}$$

Let us now define the following constants

$$\alpha \equiv \frac{l^2}{\mu k} \quad , \quad \varepsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}$$

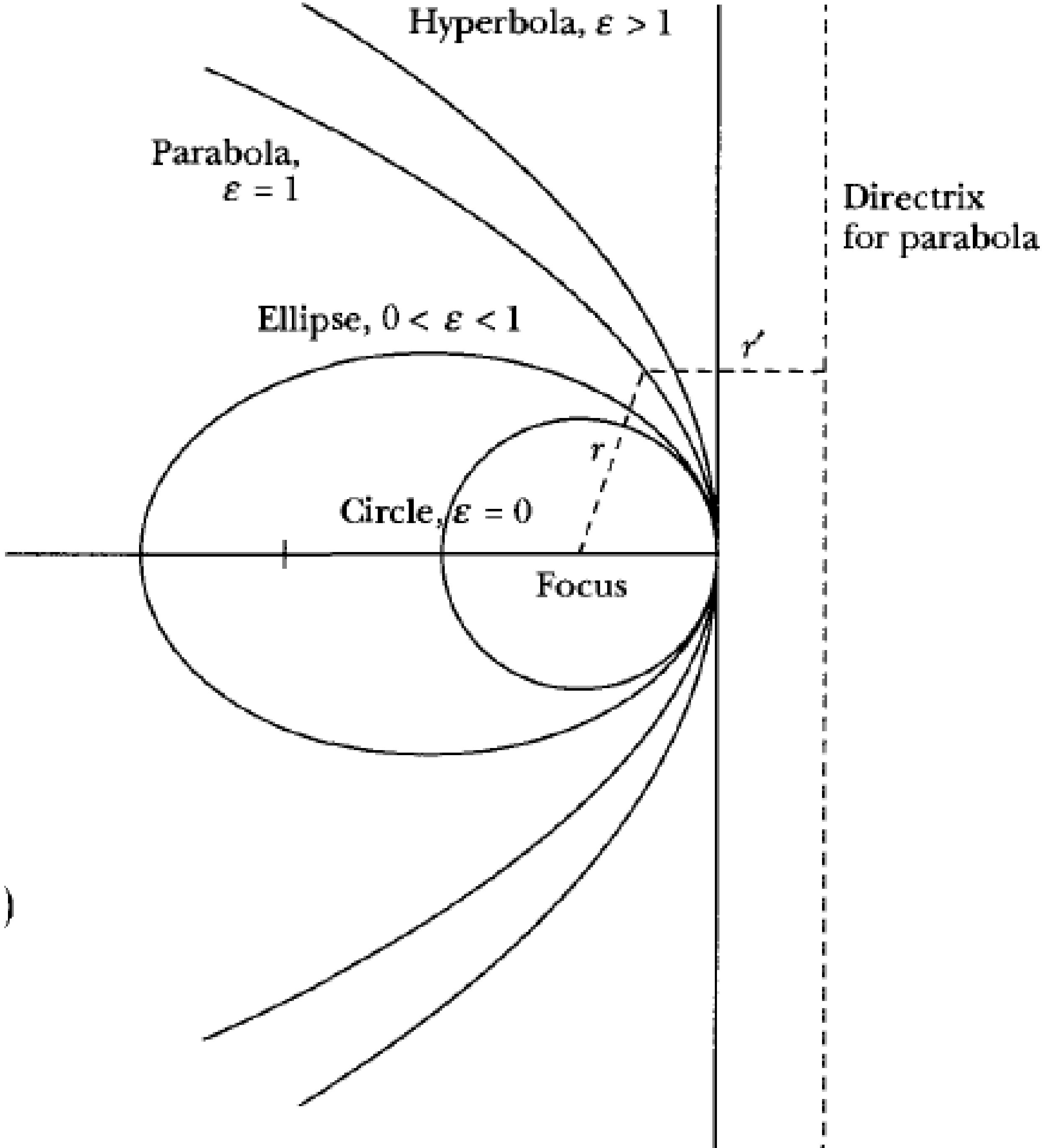
Then equation (1) can be written as:

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta$$

This is the equation of a conic section with one focus at the origin ; the quantity *ε is called the eccentricity and 2α is termed the latus rectum of the* orbit. The minimum value for *r occurs when cos θ is a maximum, i.e., for θ = 0.* Thus, the choice of *zero for the intgral constant corresponds to* measuring *θ from* r_{min}, *which position is called the pericenter;* r_{max} *corresponds* to the *apocenter.*

Various values of the eccentricity (and, hence, of the energy *E)* *classify* the orbits according to different conic sections

$\varepsilon > 1;$	$E > 0$	(hyperbola)
$\varepsilon = 1;$	$E = 0$	(parabola)
$0 < \varepsilon < 1;$	$V_{min} < E < 0$	(ellipse)
$\varepsilon = 0;$	$E = V_{min}$	(circle)
$\varepsilon < 0;$	$E < V_{min}$	(not allowed)



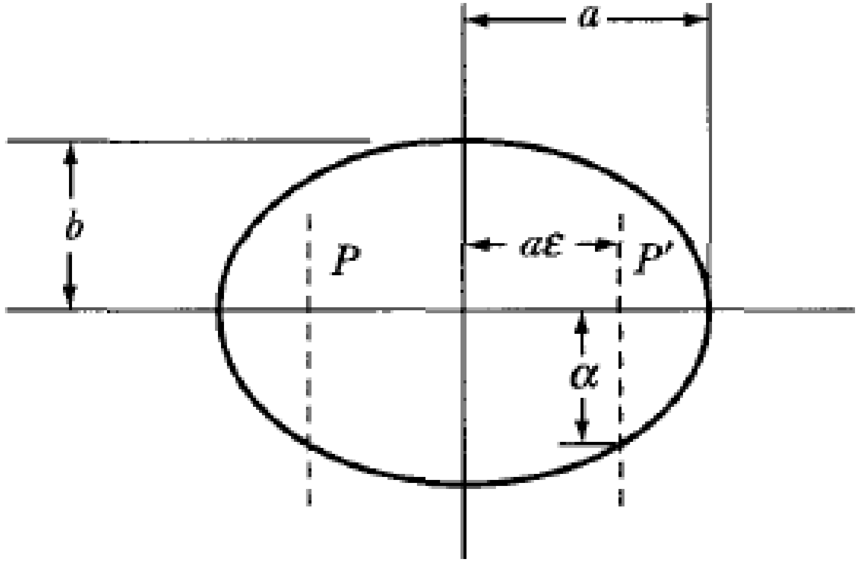
For the case of planetary motion, the orbits are ellipses with major and minor axes *{a and b, respectively) given by*

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|} \quad , \quad b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}} \qquad (2)$$

The geometry of elliptic orbits in terms of the parameters α, ε, a, and *b is* shown in Fig , *P and P' are the foci. From this diagram we see that* the apsidal distances (r_{min} and r_{max} as measured from the foci to the orbit) are given by :

$$r_{min} = a(1-\varepsilon) = \frac{\alpha}{1+\varepsilon}$$

$$r_{max} = a(1+\varepsilon) = \frac{\alpha}{1-\varepsilon}$$



In order to find the period for elliptic motion, we rewrite Equation for the areal velocity as

$$dt = \frac{2\mu}{l} dA \longrightarrow \int_0^\tau dt = \frac{2\mu}{l}\int_0^A dA \longrightarrow \tau = \frac{2\mu}{l} A$$

Since the entire area *A of the ellipse is swept out in one complete period τ,*

Now, the area of an ellipse is given by *A = πab, and using a and b from* Eqs. (2), we find

$$\tau = \frac{2\mu}{l} \cdot \pi ab = \frac{2\mu}{l} \cdot \pi \cdot \frac{k}{2|E|} \cdot \frac{l}{\sqrt{2\mu|E|}} = \pi k \sqrt{\frac{\mu}{2}} \cdot |E|^{-3/2}$$

$$\longrightarrow \quad \tau^2 = \frac{4\pi^2\mu}{k} a^3$$

This result, that the square of the period is proportional to the cube of the major axis of the elliptic orbit, is known as *Kepler's Third Law .* Note that Kepler's actual statement of his conclusion was that the squares of the periods of the planets were proportional to the cubes of the major axes of their orbits, with the same proportionality constant for all planets. In this sense, the statement is only approximately correct, since the reduced mass is different for each planet. In particular, since the gravitational force is given by

$$F(r) = -\frac{Gm_1 m_2}{r^2} = -\frac{k}{r^2}$$

we identify $k = G\, m_1\, m_2$*. Therefore, the expression for the square of the* period becomes

$$\tau^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \cong \frac{4\pi^2 a^3}{Gm_2}, \qquad m_1 \ll m_2$$

so that Kepler's statement is valid only if the mass m_1 *of a planet can be* neglected with respect to the mass m_2 *of the Sun.*

Kepler's laws can now be summarized:

I. *Planets move in elliptical orbits about the Sun with the Sun at one focus.*

II. *The area per unit time swept out by a radius vector from the Sun to a planet is constant.*

III. *The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.*

EXAMPLE 8.4

Halley's comet, which passed around the sun early in 1986, moves in a highly elliptical orbit with an eccentricity of 0.967 and a period of 76 years. Calculate its minimum and maximum distances from the Sun.

Solution. the period of motion with the semimajor axes.
Because m (Halley's comet) $\ll m_{Sun}$,

$$a = \left(\frac{G m_{Sun} \tau^2}{4\pi^2}\right)^{1/3}$$

$$= \left[\frac{\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)(1.99 \times 10^{30}\ kg)\left(76\ yr\ \frac{365\ day}{yr}\ \frac{24\ hr}{day}\ \frac{3600\ s}{hr}\right)^2}{4\pi^2}\right]^{1/3}$$

$$a = 2.68 \times 10^{12}\ m$$

Using Equation 8.44, we can determine r_{min} and r_{max}.

$$r_{min} = 2.68 \times 10^{12}\ m(1 - 0.967) = 8.8 \times 10^{10}\ m$$
$$r_{max} = 2.68 \times 10^{12}\ m(1 + 0.967) = 5.27 \times 10^{12}\ m$$

This orbit takes the comet inside the path of Venus, almost to Mercury's orbit, and out past even the orbit of Neptune and sometimes even to the moderately eccentric orbit of Pluto.