Vectors

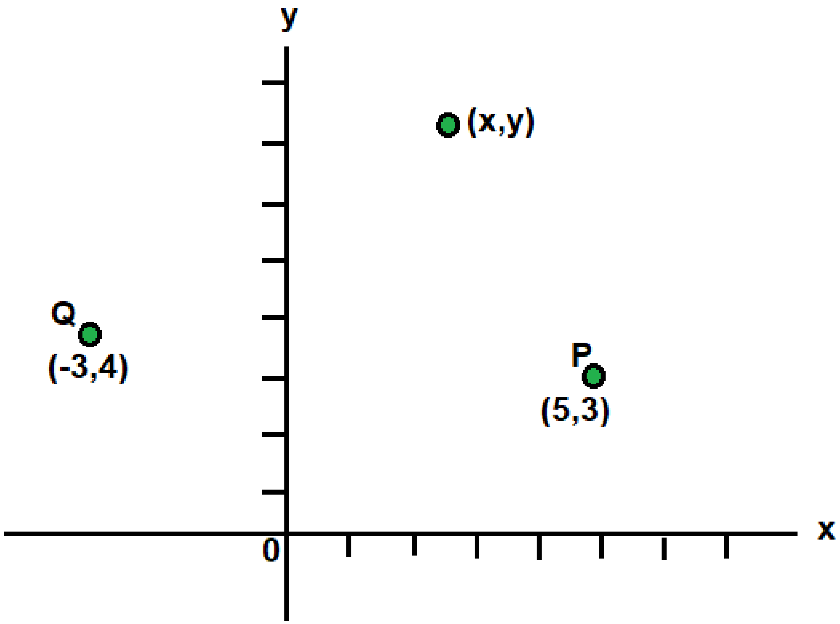
Chapter 3

3.1 Coordinate Systems

- Describe the position of a point in space
- Coordinate system consists of
  - a fixed reference point called the origin
  - specific axes with scales and labels
  - instructions on how to label a point relative to the origin and the axes

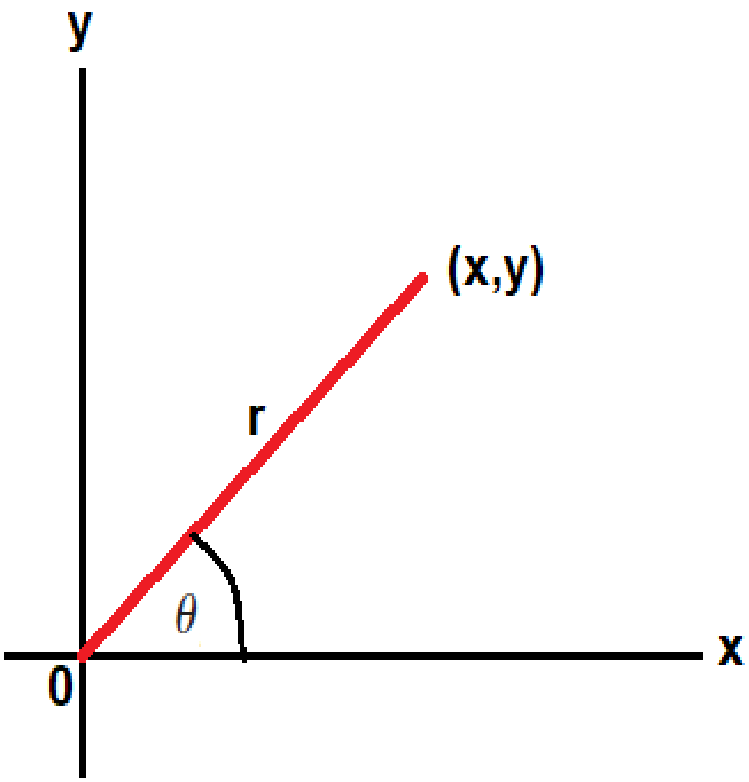
Cartesian Coordinate System

- Also called rectangular coordinate system
- *x*- and *y*- axes intersect at the origin
- Points are labeled (*x*, *y*)



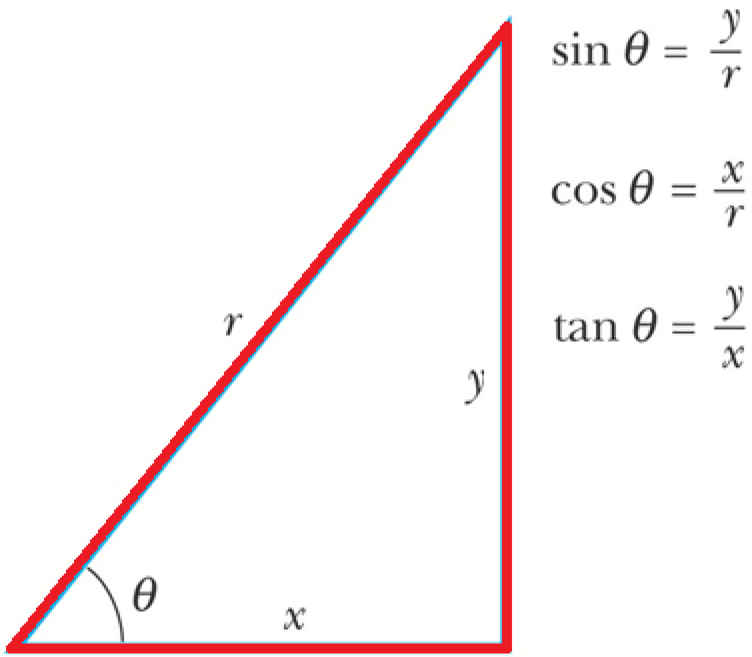
Polar Coordinate System

- Origin and reference line are noted
- Point is distance *r* from the origin in the direction of angle θ, ccw from reference line
- Points are labeled (r, θ)



Polar to Cartesian Coordinates

- Based on forming a right triangle from r and θ
- $x = r\cos\theta$
- $y = r\sin\theta$



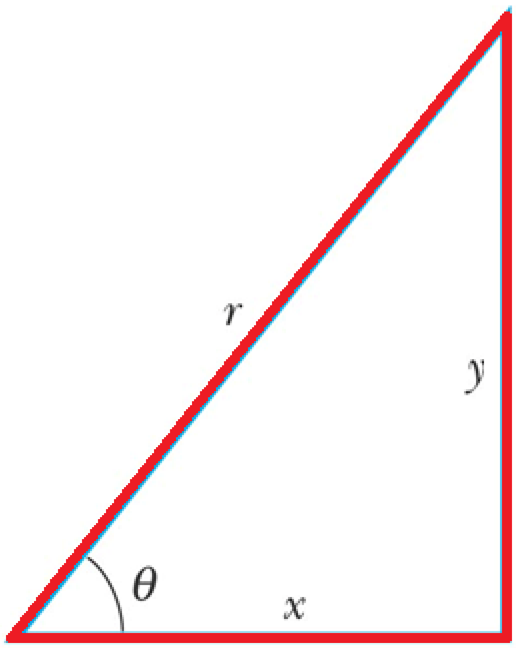
Cartesian to Polar Coordinates

- r is the hypotenuse and θ an angle

$$\tan\theta = \frac{y}{x}$$

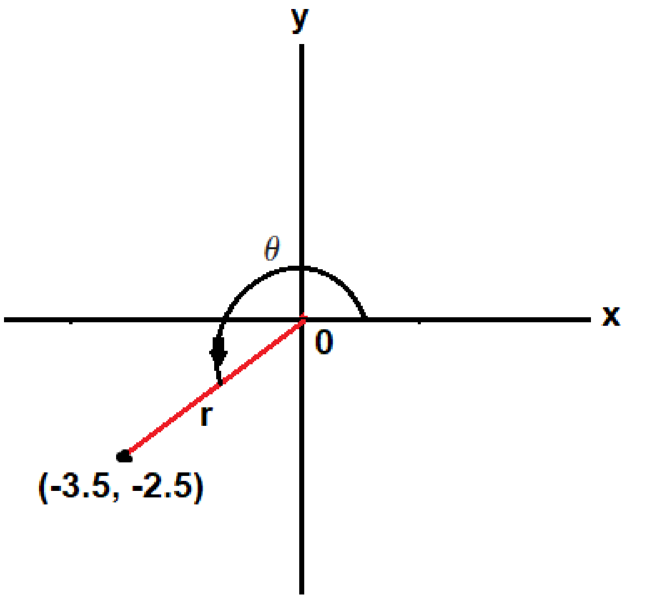
$$r = \sqrt{x^2 + y^2}$$

- θ must be ccw from positive x axis for these equations to be valid



Example 3.1

- The Cartesian coordinates of a point in the *xy* plane are $(x,y) = (-3.50, -2.50)$ m, as shown in the figure. Find the polar coordinates of this point.



- **Solution:**

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3.50 \text{ m})^2 + (-2.50 \text{ m})^2} = 4.30 \text{ m}$$

and

$$\tan\theta = \frac{y}{x} = \frac{-2.50 \text{ m}}{-3.50 \text{ m}} = 0.714$$

$$\theta = 216°$$

3.2 Vector and Scalar Quantities

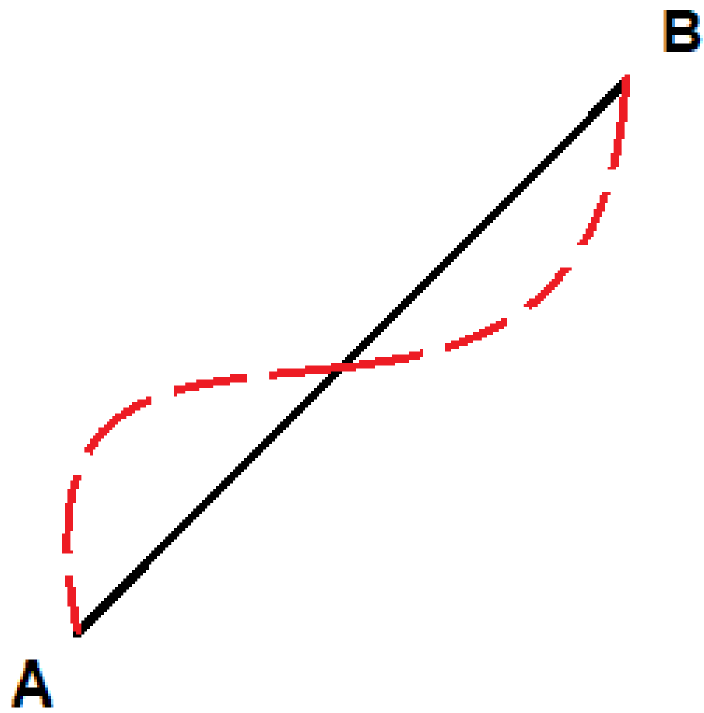
- A *scalar quantity* is completely specified by a single value with an appropriate unit and has no direction.
- A *vector quantity* is completely described by a number and appropriate units plus a direction.

Vector Notation

- When handwritten, use an arrow: $\vec{\mathrm{A}}$
- When printed, will be in bold print: **A**
- When dealing with just the magnitude of a vector in print, an italic letter will be used: *A* or |**A**|
- The magnitude of the vector has physical units
- The magnitude of a vector is always a positive number

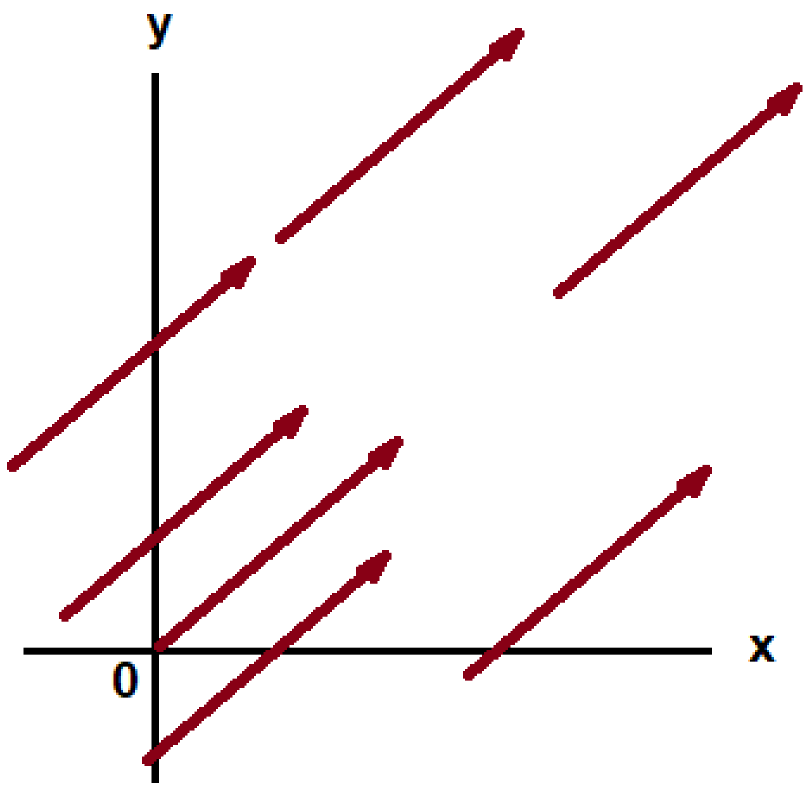
Vector Example

- A particle travels from A to B along the path shown by the dotted red line
  - This is the ***distance*** traveled and is a scalar
- The ***displacement*** is the solid line from A to B
  - The displacement is independent of the path taken between the two points
  - Displacement is a vector



3.3 Some Properties of Vectors

- Two vectors are ***equal*** if they have the same magnitude and the same direction
- **A** = **B** if *A* = *B* and they point along parallel lines
- All of the vectors shown are equal



Adding Vectors

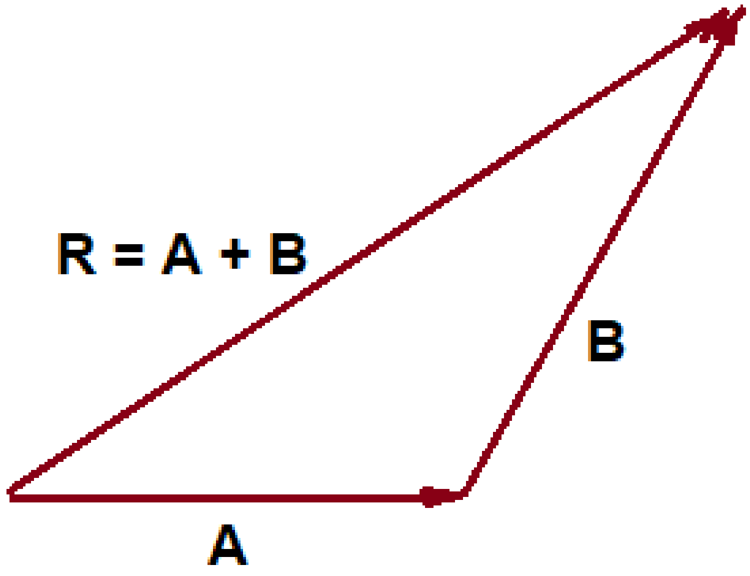
- When adding vectors, their directions must be taken into account
- Units must be the same
- Graphical Methods
  - Use scale drawings
- Algebraic Methods
  - More convenient

Adding Vectors Graphically

- Choose a scale
- Draw the first vector with the appropriate length and in the direction specified, with respect to a coordinate system
- Draw the next vector with the appropriate length and in the direction specified, with respect to a coordinate system whose origin is the end of vector **A** and parallel to the coordinate system used for **A**

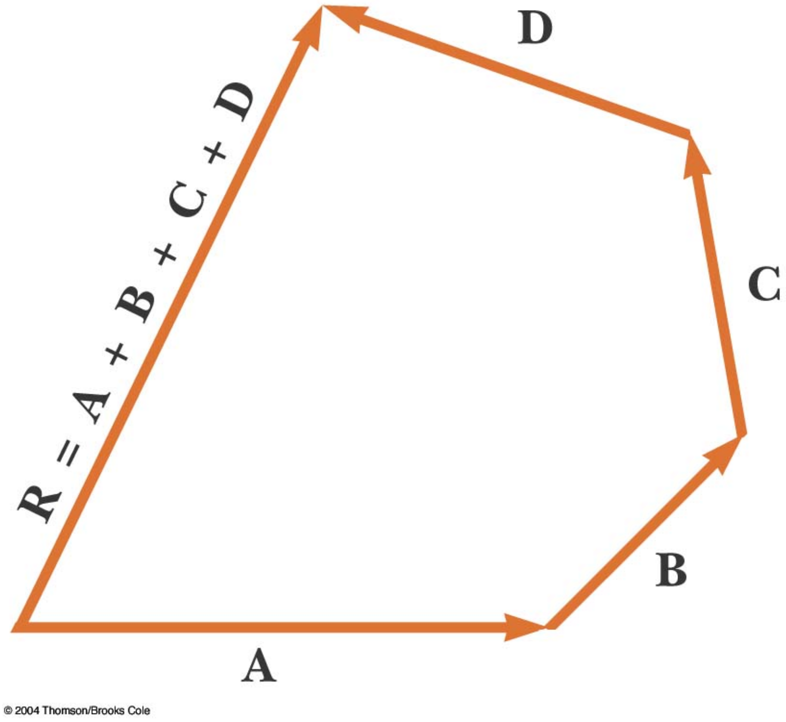
Adding Vectors Graphically, cont.

- Continue drawing the vectors “tip-to-tail”
- The resultant is drawn from the origin of **A** to the end of the last vector
- Measure the length of **R** and its angle
  - Use the scale factor to convert length to actual magnitude



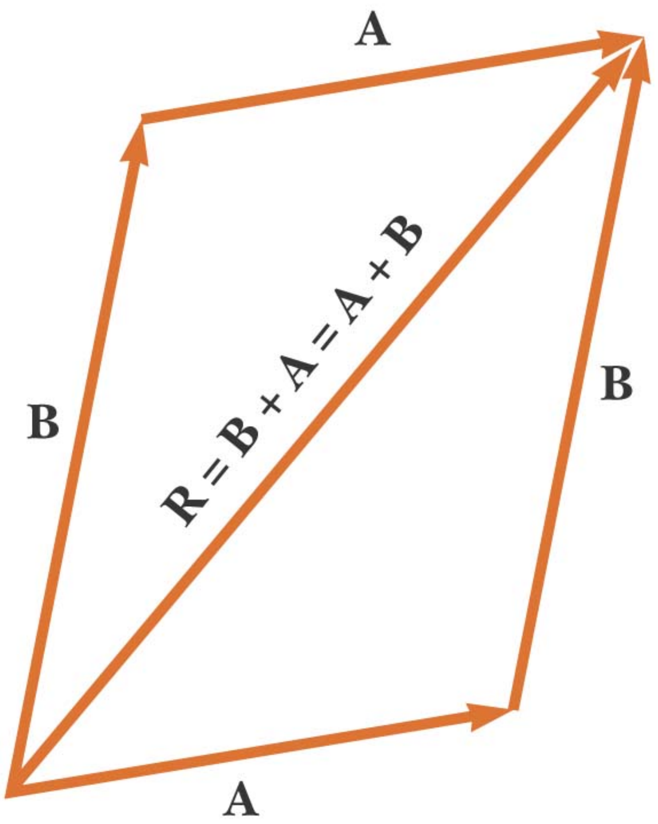
Adding Vectors Graphically, final

- When you have many vectors, just keep repeating the process until all are included
- The resultant is still drawn from the origin of the first vector to the end of the last vector



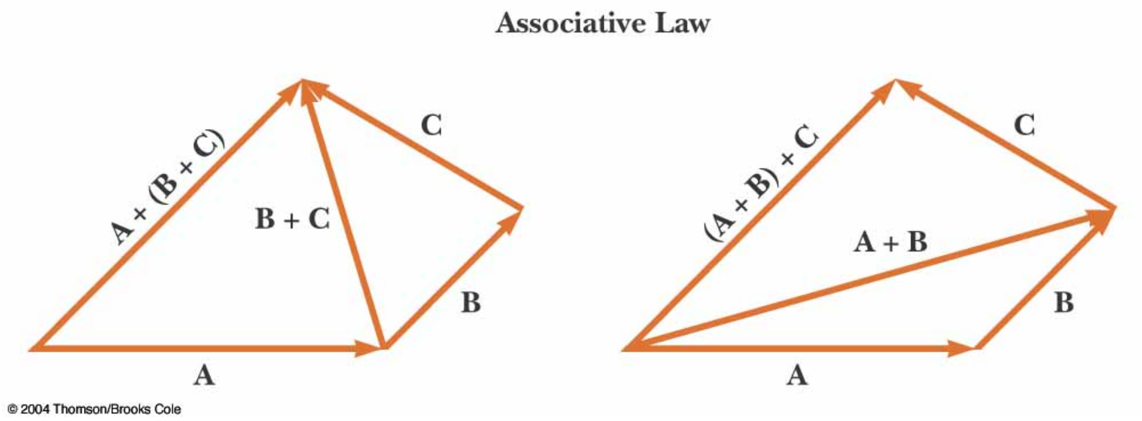
Adding Vectors, Rules

- When two vectors are added, the sum is independent of the order of the addition.
  - This is the *commutative law of addition*
  - **A** + **B** = **B** + **A**



Adding Vectors, Rules cont.

- When adding three or more vectors, their sum is independent of the way in which the individual vectors are grouped
  - This is called the ***Associative Property of Addition***
  - **(A + B) + C = A + (B + C)**



Adding Vectors, Rules final

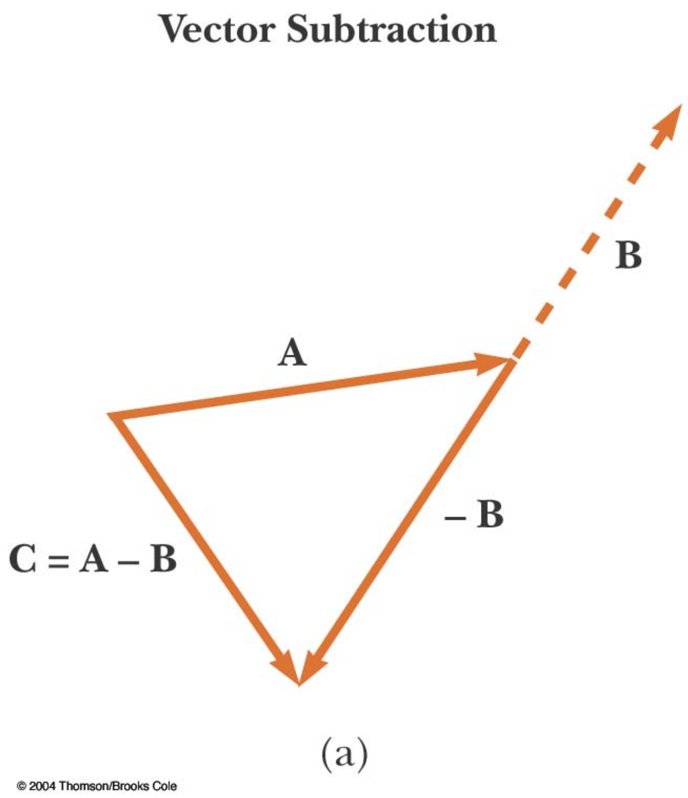
- When adding vectors, all of the vectors must have the same units
- All of the vectors must be of the same type of quantity
  - For example, you cannot add a displacement to a velocity

Negative of a Vector

- The negative of a vector is defined as the vector that, when added to the original vector, gives a resultant of zero
  - Represented as –**A**
  - **A** + (-**A**) = 0
- The negative of the vector will have the same magnitude, but point in the opposite direction

Subtracting Vectors

- Special case of vector addition
- If **A** – **B**, then use **A**+(-**B**)
- Continue with standard vector addition procedure

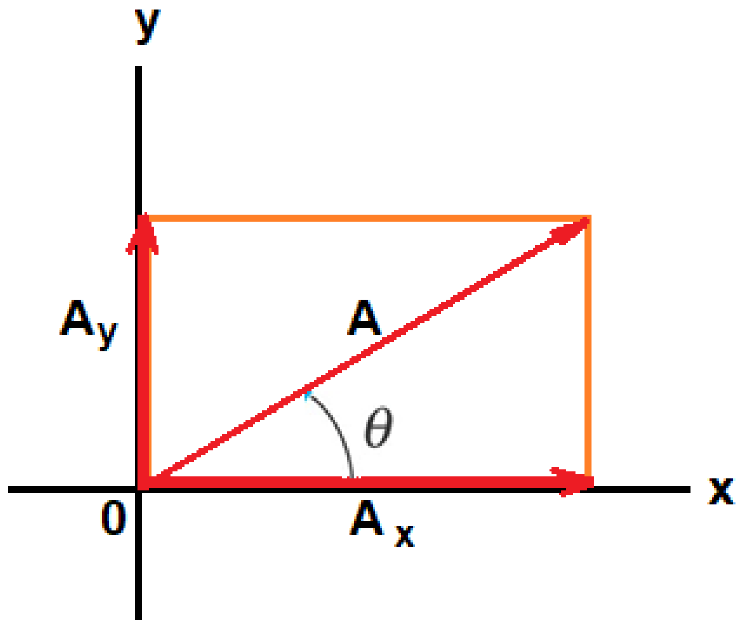


Multiplying or Dividing a Vector by a Scalar

- The result of the multiplication or division is a vector
- The magnitude of the vector is multiplied or divided by the scalar
- If the scalar is positive, the direction of the result is the same as of the original vector
- If the scalar is negative, the direction of the result is opposite that of the original vector

3.4 Components of a Vector and Unit Vectors

- A **component** is a part
- It is useful to use **rectangular components**
  - These are the projections of the vector along the x- and y-axes



Vector Component Terminology

- $\mathbf{A}_x$ and $\mathbf{A}_y$ are the ***component vectors*** of **A**
  - They are vectors and follow all the rules for vectors
- A_x and A_y are scalars, and will be referred to as the ***components*** of **A**

Components of a Vector, 2

- The x-component of a vector is the projection along the x-axis

$$A_x = A\cos\theta$$

- The y-component of a vector is the projection along the y-axis

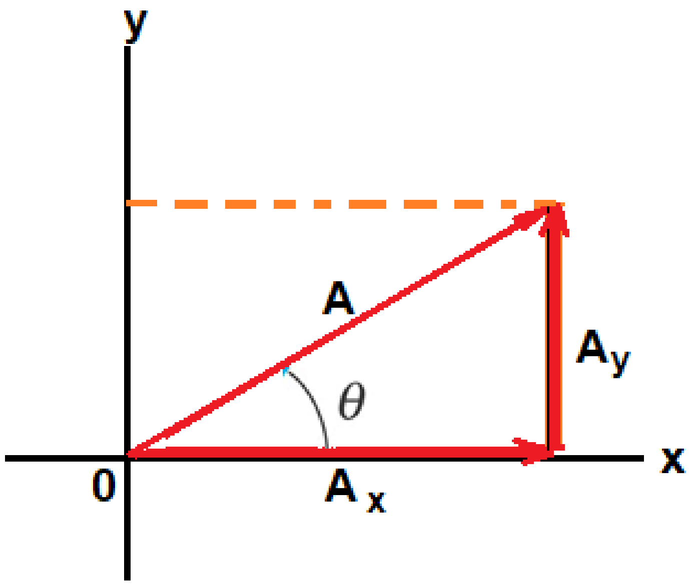
$$A_y = A\sin\theta$$

- Then,

$$\mathbf{A} = A_x + A_y$$

Components of a Vector, 3

- The *y*-component is moved to the end of the *x*-component
- This is due to the fact that any vector can be moved parallel to itself without being affected
  - This completes the triangle



$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{A_y}{A_x}$$

- The components can be positive or negative and will have the same units as the original vector
- The signs of the components will depend on the angle
- The component equations

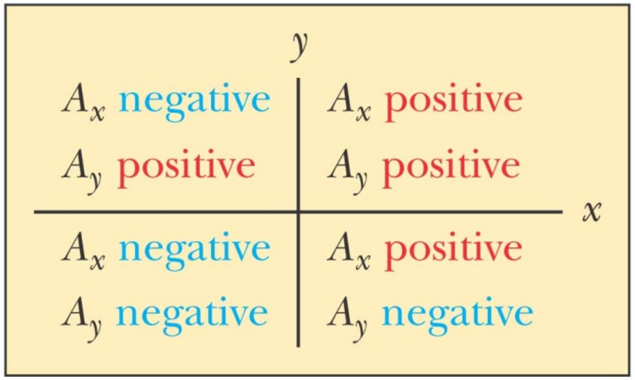
$$(A_x = A\cos\theta \text{ and } A_y = A\sin\theta)$$

apply only when the angle is measured with respect to the *x*-axis (preferably ccw from the positive *x*-axis).

- The resultant angle

$$(\tan\theta = A_y / A_x)$$

gives the angle with respect to the *x*-axis.



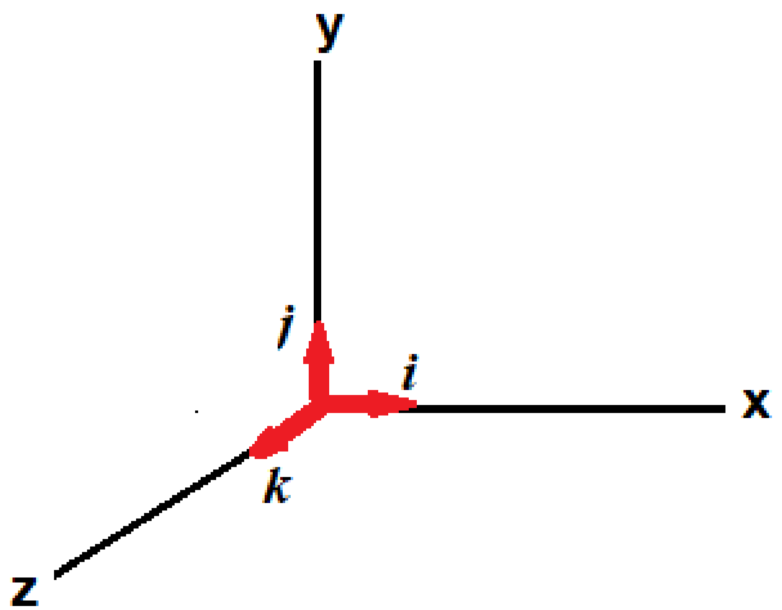
Unit Vectors

- A ***unit vector*** is a dimensionless vector with a magnitude of exactly 1.
- The symbols

  $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$

represent unit vectors

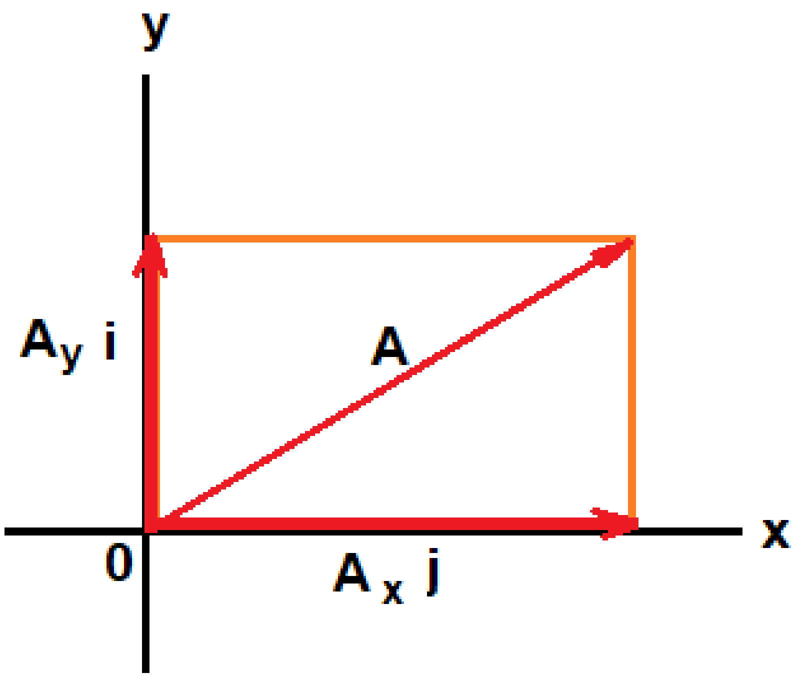
- They form a set of mutually perpendicular vectors



Unit Vectors in Vector Notation

- $\mathbf{A_x}$ is the same as $A_x\,\hat{\mathbf{i}}$ and $\mathbf{A_y}$ is the same as $A_y\,\hat{\mathbf{j}}$ etc.
- The complete vector can be expressed as

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$$



Adding Vectors Using Unit Vectors

- Using **R** = **A** + **B**
- Then

$$\mathbf{R} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}}\right)$$

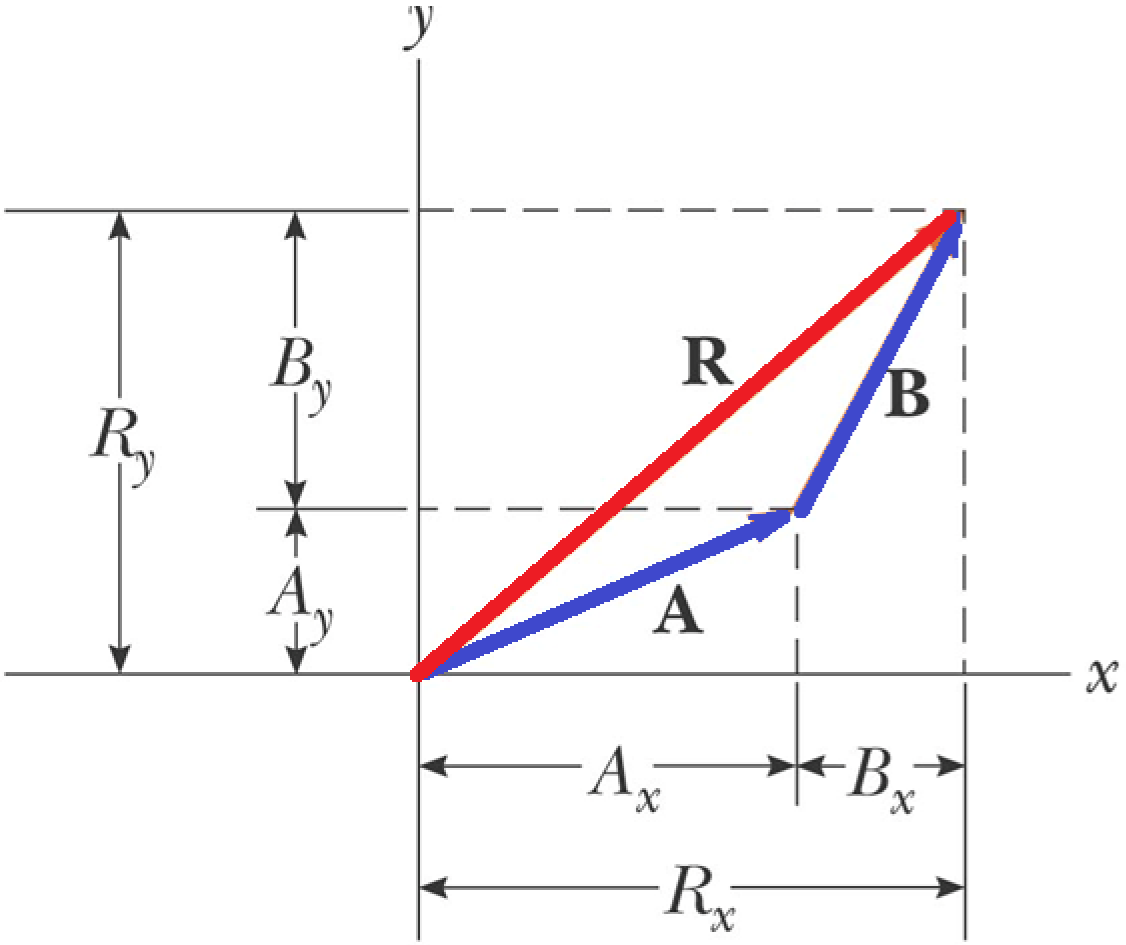
$$\mathbf{R} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}}$$

$$\mathbf{R} = R_x + R_y$$

- and so $R_x = A_x + B_x$ and $R_y = A_y + B_y$

$$R = \sqrt{R_x^2 + R_y^2} \qquad \theta = \tan^{-1}\frac{R_y}{R_x}$$

Adding Vectors with Unit Vectors



Adding Vectors Using Unit Vectors – Three Directions

- Using **R** = **A** + **B**

$$\mathbf{R} = \left(A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}\right) + \left(B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}\right)$$

$$\mathbf{R} = \left(A_x + B_x\right)\hat{\mathbf{i}} + \left(A_y + B_y\right)\hat{\mathbf{j}} + \left(A_z + B_z\right)\hat{\mathbf{k}}$$

$$\mathbf{R} = R_x + R_y + R_z$$

- $R_x = A_x + B_x$, $R_y = A_y + B_y$ and $R_z = A_z + B_z$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \qquad \theta_x = \tan^{-1}\frac{R_x}{R}$$

etc.

- Example : **The Sum of Two Vectors**

Find the sum of two vectors **A** and **B** lying in the xy- plane and given by:

$$\vec{\mathbf{A}} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}})\ \text{m} \qquad \text{and} \qquad \vec{\mathbf{B}} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})\ \text{m}$$

- Solution:

$$A_x = 2.0\ \text{m and } A_y = 2.0\ \text{m.}$$

$$B_x = 2.0\ \text{m and } B_y = -4.0\ \text{m}$$

$$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} = (2.0 + 2.0)\hat{\mathbf{i}}\ \text{m} + (2.0 - 4.0)\hat{\mathbf{j}}\ \text{m}$$

$$R_x = 4.0\ \text{m} \qquad R_y = -2.0\ \text{m}$$

the magnitude of $\vec{\mathbf{R}}$:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(4.0\ \text{m})^2 + (-2.0\ \text{m})^2} = \sqrt{20}\ \text{m} = 4.5\ \text{m}$$

the direction of $\vec{\mathbf{R}}$:

$$\tan\theta = \frac{R_y}{R_x} = \frac{-2.0\ \text{m}}{4.0\ \text{m}} = -0.50$$

- Example : **The Resultant Displacement**

A particle undergoes three consecutive displacements:

$$\Delta\vec{\mathbf{r}}_1 = (15\hat{\mathbf{i}} + 30\hat{\mathbf{j}} + 12\hat{\mathbf{k}})\ \text{cm},$$

$$\Delta\vec{\mathbf{r}}_2 = (23\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 5.0\hat{\mathbf{k}})\ \text{cm}$$

$$\Delta\vec{\mathbf{r}}_3 = (-13\hat{\mathbf{i}} + 15\hat{\mathbf{j}})\ \text{cm}$$

Find the components of the resultant displacement and its magnitude.

- Solution:

$$\Delta\vec{\mathbf{r}} = \Delta\vec{\mathbf{r}}_1 + \Delta\vec{\mathbf{r}}_2 + \Delta\vec{\mathbf{r}}_3$$

$$= (15 + 23 - 13)\hat{\mathbf{i}}\ \text{cm} + (30 - 14 + 15)\hat{\mathbf{j}}\ \text{cm} + (12 - 5.0 + 0)\hat{\mathbf{k}}\ \text{cm}$$

$$= (25\hat{\mathbf{i}} + 31\hat{\mathbf{j}} + 7.0\hat{\mathbf{k}})\ \text{cm}$$

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

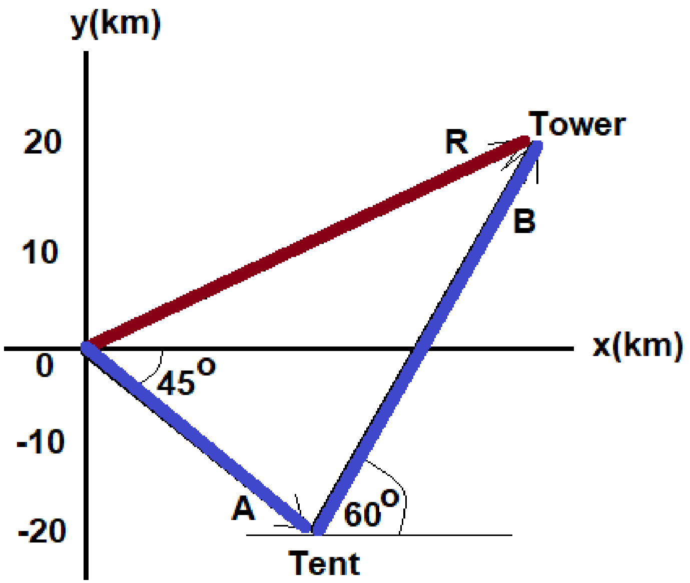
$$= \sqrt{(25\ \text{cm})^2 + (31\ \text{cm})^2 + (7.0\ \text{cm})^2} = 40\ \text{cm}$$

- Example : **Taking a Hike**

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60.0° north of east, at which point she discovers a forest ranger's tower

(A) Determine the components of the hiker's displacement for each day.

(B) Determine the components of the hiker's resultant displacement ***R*** for the trip. Find an expression for ***R*** in terms of unit vectors



- Solution:

(A) - the components of A

$$A_x = A\cos(-45.0°) = (25.0\text{ km})(0.707) = 17.7\text{ km}$$

$$A_y = A\sin(-45.0°) = (25.0\text{ km})(-0.707) = -17.7\text{ km}$$

the components of B

$$B_x = B\cos 60.0° = (40.0\text{ km})(0.500) = 20.0\text{ km}$$

$$B_y = B\sin 60.0° = (40.0\text{ km})(0.866) = 34.6\text{ km}$$

(B) -

$$R_x = A_x + B_x = 17.7\text{ km} + 20.0\text{ km} = 37.7\text{ km}$$

$$R_y = A_y + B_y = -17.7\text{ km} + 34.6\text{ km} = 16.9\text{ km}$$

$$\vec{\mathbf{R}} = (37.7\hat{\mathbf{i}} + 16.9\hat{\mathbf{j}})\text{ km}$$

Problem Solving Strategy

- Select a coordinate system
  - Try to select a system that minimizes the number of components you need to deal with
- Draw a sketch of the vectors
  - Label each vector
- Find the x and y components of each vector and the x and y components of the resultant vector
  - Find z components if necessary
- Use the Pythagorean theorem to find the magnitude of the resultant and the tangent function to find the direction
  - Other appropriate trig functions may be used

Problems

- Problem (1) :
- Two points in a plane have polar coordinates (2.50 m, 30.0°) and (3.80 m, 120.0°). Determine (a) the Cartesian coordinates of these points and (b) the distance between them.
- Solution:

(a) $x = r\cos\theta$ and $y = r\sin\theta$, therefore

$$x_1 = (2.50\text{ m})\cos 30.0°,\quad y_1 = (2.50\text{ m})\sin 30.0°,\text{ and}$$

$$(x_1, y_1) = \boxed{(2.17,\ 1.25)\text{ m}}$$

$$x_2 = (3.80\text{ m})\cos 120°,\quad y_2 = (3.80\text{ m})\sin 120°,\text{ and}$$

$$(x_2, y_2) = \boxed{(-1.90,\ 3.29)\text{ m}}.$$

(b) $$d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{16.6 + 4.16} = \boxed{4.55\text{ m}}$$

- Problems (2):
- Two points in the *xy* plane have Cartesian coordinates (2.00, –4.00) m and (–3.00, 3.00) m. Determine (a) the distance between these points and (b) their polar coordinates.
- Solution:

(a) $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2.00 - [-3.00])^2 + (-4.00 - 3.00)^2}$$

$$d = \sqrt{25.0 + 49.0} = \boxed{8.60\ \text{m}}$$

(b) $$r_1 = \sqrt{(2.00)^2 + (-4.00)^2} = \sqrt{20.0} = \boxed{4.47\ \text{m}}$$

$$\theta_1 = \tan^{-1}\left(-\frac{4.00}{2.00}\right) = \boxed{-63.4°}$$

$$r_2 = \sqrt{(-3.00)^2 + (3.00)^2} = \sqrt{18.0} = \boxed{4.24\ \text{m}}$$

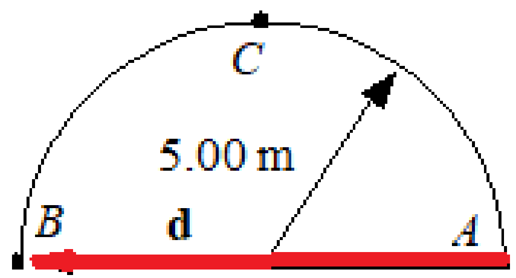
$$\theta_2 = \boxed{135°}$$ measured from the $+x$ axis.

- Problems (3):
- A skater glides along a circular path of radius 5.00 m. If he coasts around one half of the circle, find (a) the magnitude of the displacement vector and (b) how far the person skated. (c) What is the magnitude of the displacement if he skates all the way around the circle?
- Solution:

(a) $|\mathbf{d}| = |-10.0\hat{\mathbf{i}}| = \boxed{10.0 \text{ m}}$ since the displacement is in a straight line from point A to point B.

(b) The actual distance skated is not equal to the straight-line displacement. The distance follows the curved path of the semi-circle (ACB).

$$s = \frac{1}{2}(2\pi r) = 5\pi = \boxed{15.7 \text{ m}}$$



(c) If the circle is complete, **d** begins and ends at point A.

Hence, $|\mathbf{d}| = \boxed{0}$.

- Problems (4):
- Obtain expressions in component form for the position vectors having polar coordinates (a) 12.8 m, 150°; (b) 3.30 cm, 60.0°; (c) 22.0 in., 215°.
- Solution:

$x = r\cos\theta$ and $y = r\sin\theta$, therefore:

(a) $x = 12.8\cos 150°$, $y = 12.8\sin 150°$, and $(x, y) = (-11.1\hat{\mathbf{i}} + 6.40\hat{\mathbf{j}})$ m

(b) $x = 3.30\cos 60.0°$, $y = 3.30\sin 60.0°$, and $(x, y) = (1.65\hat{\mathbf{i}} + 2.86\hat{\mathbf{j}})$ cm

(c) $x = 22.0\cos 215°$, $y = 22.0\sin 215°$, and $(x, y) = (-18.0\hat{\mathbf{i}} - 12.6\hat{\mathbf{j}})$ in

- Problems (5):
- Consider the two $\mathbf{A} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ vectors $\mathbf{B} = -\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$ and . Calculate (a) **A** + **B**, (b) **A** – **B**, (c) |**A** + **B**|, (d) |**A** – **B**|, and (e) the directions of **A** + **B** and **A** – **B**
- Solution:

(a) $(\mathbf{A}+\mathbf{B}) = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}) + (-\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) = \boxed{2\hat{\mathbf{i}} - 6\hat{\mathbf{j}}}$, (c) $|\mathbf{A}+\mathbf{B}| = \sqrt{2^2+6^2} = \boxed{6.32}$

(b) $(\mathbf{A}-\mathbf{B}) = (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}) - (-\hat{\mathbf{i}} - 4\hat{\mathbf{j}}) = \boxed{4\hat{\mathbf{i}} + 2\hat{\mathbf{j}}}$, (d) $|\mathbf{A}-\mathbf{B}| = \sqrt{4^2+2^2} = \boxed{4.47}$

(e)
$$\theta_{|\mathbf{A}+\mathbf{B}|} = \tan^{-1}\left(-\frac{6}{2}\right) = -71.6^\circ = \boxed{288^\circ}$$

$$\theta_{|\mathbf{A}-\mathbf{B}|} = \tan^{-1}\left(\frac{2}{4}\right) = \boxed{26.6^\circ}$$