Hamilton s Principle—Lagrangian and

Hamiltonian Dynamics

Ch. 7

7.1 Introduction

In solving a problem by using the Newtonian procedure, however, it is necessary to know *all of the forces since* the quantity F which appears in the fundamental equation is the *total* force acting on a body.

in particular situations it may be difficult or even impossible to obtain explicit expressions for the forces of constraint.

In order to circumvent some of the practical difficulties which arise in attempts to apply Newton's equations to particular problems, alternative procedures may be developed.

Such a method is contained in *Hamilton's Principle and the equations of motion which result* from the application of this principle are called *Lagrange's equations.*

7.2 Hamilton's Principle

Hamilton's Principle may be stated as follows :

Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energies.

$$\delta \int_{t_1}^{t_2} (T - U)\, dt = 0$$

we shall confine our attention to *conservative systems. In rectangular coordinate system. Such that:*

$$T = T(\dot{x}_i); \qquad U = U(x_i)$$

If we define the difference of these quantities to be

$$L \equiv T - U$$
$$= L(x_i, \dot{x}_i)$$

Hamilton's Principle becomes

$$\delta \int_{t_1}^{t_2} L(x_i, \dot{x}_i)\, dt = 0$$

By the same manner as in ch.6 , we can drive the following equation

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0, \qquad i = 1, 2, 3$$

These are the *Lagrange equations of motion for the particle and the quantity* *L* is called the *Lagrange function or Lagrangian for the particle.*

As an example, let us obtain the Lagrange equation of motion for the one-dimensional harmonic oscillator. In this case the Lagrangian is

$$L = T - U = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2$$

By appling Lagrange equation, we get directly

$$m\ddot{x} + kx = 0$$

which is identical with the Newtonian equation of motion.

Another example as in the case of the plane pendulum *we have for the* Lagrangian function

$$L = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

We now treat *θ as if it were a rectangular coordinate and apply* the Lagrange equation of motion, we obtain :

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

which again is identical with the Newtonian equation.

important characteristic of the method employed in the two simple examples above is the fact that nowhere in the calculations did there enter any statement regarding *force. Thus, the equations of motion* were obtained only by specifying certain properties associated *with the particle (the kinetic and potential energies), and without the necessity of* explicitly taking into account the fact that there was an external agency acting *on the particle (the force).*

7.3 Generalized coordinates

In order to specify the state of a system, it is necessary to use n radius vectors. Since each radius vector consists of a triple of numbers **3n** quantities must be specified in order to describe the positions of all the particles.

If there exist equations of constraint which relate some of these coordinates to others

In fact, if there are **m** equations of constraint, then **(3n — m)** coordinates are independent, and the system is said to possess **s =(3n — m)** degrees of freedom.

it is possible to choose any **s parameters**, as long as they completely specify the state of the system. These s quantities need not even have the dimensions of length.

A set of independent generalized coordinates whose number equals the number **s** of degrees of freedom of the system and which are not restricted by the constraints will be called a proper set of generalized coordinates.

In addition to the generalized coordinates, we may define a set of quantities which consists of the time derivatives of the q_i, we call them the generalized velocities.

the equations connecting the $\mathbf{x}_{\alpha i}$ and the $\boldsymbol{q}_i$ explicitly contain the time, then the set of transformation equations is given by

$$x_{\alpha,i} = x_{\alpha,i}(q_j, t), \qquad \alpha = 1, 2, \ldots, n$$

$$\dot{x}_{\alpha,i} = \dot{x}_{\alpha,i}(q_j, \dot{q}_j, t) \qquad i = 1, 2, 3$$

$$j = 1, 2, \ldots, s$$

the inverse transformations is

$$q_j = q_j(x_{\alpha,i}, t)$$

$$\dot{q}_j = \dot{q}_j(x_{\alpha,i}, \dot{x}_{\alpha,i}, t)$$

there are also m = 3n — s equations of constraint of the form

$$f_l = f_l(x_{\alpha,i}, t), \qquad l = 1, 2, \ldots, m$$

EXAMPLE 7.1

Find a suitable set of generalized coordinates for a point particle moving on the surface of a hemisphere of radius R whose center is at the origin.

Solution. Because the motion always takes place on the surface, we have

$$x^2 + y^2 + z^2 - R^2 = 0, \quad z \geq 0 \tag{7.10}$$

Let us choose as our generalized coordinates the cosines of the angles between the x-, y-, and z-axes and the line connecting the particle with the origin.

Therefore,

$$q_1 = \frac{x}{R}, \qquad q_2 = \frac{y}{R}, \qquad q_3 = \frac{z}{R} \tag{7.11}$$

But the sum of the squares of the direction cosines of a line equals unity. Hence,

$$q_1^2 + q_2^2 + q_3^2 = 1 \tag{7.12}$$

This set of q_j does not constitute a proper set of generalized coordinates, because we can write q_3 as a function of q_1 and q_2:

$$q_3 = \sqrt{1 - q_1^2 - q_2^2} \tag{7.13}$$

We may, however, choose $q_1 = x/R$ and $q_2 = y/R$ as proper generalized coordinates, and these quantities, together with the equation of constraint (Equation 7.13)

$$z = \sqrt{R^2 - x^2 - y^2} \tag{7.14}$$

are sufficient to uniquely specify the position of the particle. This should be an obvious result, because only two coordinates (e.g., latitude and longitude) are necessary to specify a point on the surface of a sphere. But the example illustrates the fact that the equations of constraint can always be used to reduce a trial set of coordinates to a proper set of generalized coordinates.

EXAMPLE 7.2

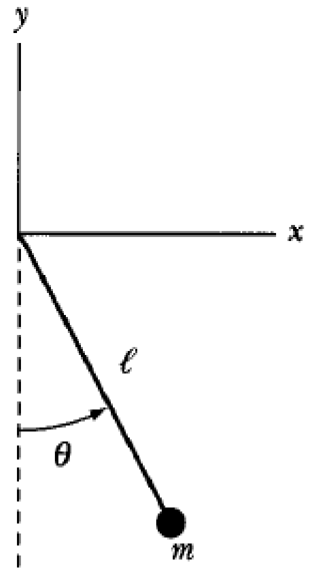
Use the (x, y) coordinate system of Figure 7-1 to find the kinetic energy T, potential energy U, and the Lagrangian L for a simple pendulum (length ℓ, mass bob m) moving in the x, y plane. Determine the transformation equations from the (x, y) rectangular system to the coordinate θ. Find the equation of motion.

Solution. We have already examined this general problem in Sections 4.4 and 7.1. When using the Lagrangian method, it is often useful to begin with rectangular coordinates and transform to the most obvious system with the simplest generalized coordinates. In this case, the kinetic and potential energies and the Lagrangian become

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2$$

$$U = mgy$$

$$L = T - U = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 - mgy$$



Example 7.2. A simple pendulum of length ℓ and bob of mass m.

Inspection of Figure 7-1 reveals that the motion can be better described by using θ and $\dot{\theta}$. Let's transform x and y into the coordinate θ and then find L in terms of θ.

$$x = \ell \sin \theta$$
$$y = -\ell \cos \theta$$

We now find for $\dot{x}$ and $\dot{y}$

$$\dot{x} = \ell\dot{\theta} \cos \theta$$
$$\dot{y} = \ell\dot{\theta} \sin \theta$$
$$L = \frac{m}{2}(\ell^2\dot{\theta}^2 \cos^2 \theta + \ell^2\dot{\theta}^2 \sin^2 \theta) + mg\ell \cos \theta = \frac{m}{2}\ell^2\dot{\theta}^2 + mg\ell \cos \theta$$

The only generalized coordinate in the case of the pendulum is the angle θ, and we have expressed the Lagrangian in terms of θ by following a simple procedure of finding L in terms of x and y, finding the transformation equations, and then inserting them into the expression for L. If we do as we did in the previous section and treat θ *as if it were a rectangular coordinate,* we can find the equation of motion as follows:

$$\frac{\partial L}{\partial \theta} = -mg\ell \sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta}$$

We insert these relations into Equation 7.4 to find the same equation of motion as found previously.

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

7.4 Lagrange's Equations of Motion in Generalized Coordinates

-we may now restate Hamilton's Principle as follows:

-Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval, the actual path followed is that which minimizes the time integral of the Lagrangian function for the system.

-the Lagrangian must be invariant with respect to coordinate transformations. We are therefore assured that no matter what generalized coordinates are chosen for the description of a system, the Lagrangian will have the same value for a given condition of the system.

-we express the Lagrangian in terms of the $x_{\alpha,i}$ and $\dot{x}_{\alpha,i}$ or the q_j and $\dot{q}_j$:

$$L = T(\dot{x}_{\alpha,i}) - U(x_{\alpha,i})$$
$$= T(q_j, \dot{q}_j, t) - U(q_j, t)$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0, \qquad j = 1, 2, \ldots, s$$

-the validity of Lagrange's equations requires the following two conditions :

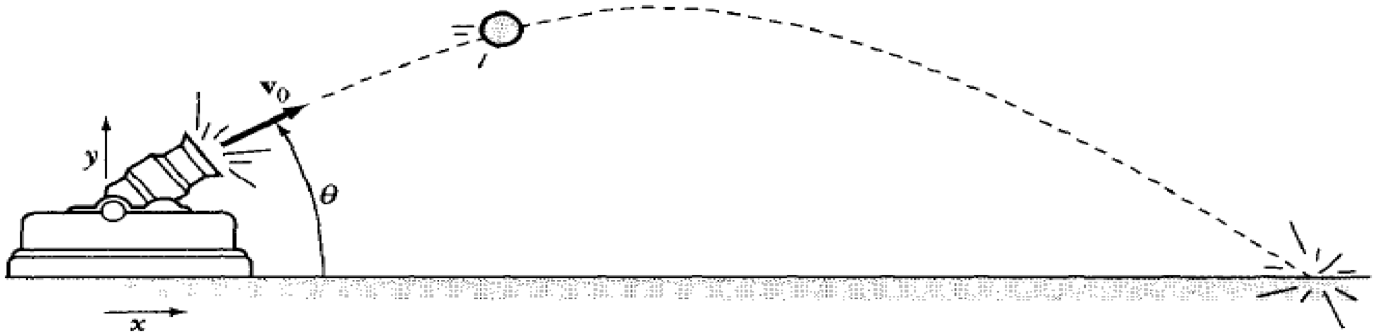
1- As stated earlier, we shall consider only the motion of systems subject to conservative forces. Such forces may always be derived from potential functions,

2- the equations of constraint must be relations that connect the coordinates of the particles and may be functions of the time

EXAMPLE 7.3

Consider the case of projectile motion under gravity in two dimensions Find the equations of motion in both Cartesian and polar coordinates.

-**Solution :**



- **First in Cartesian coordinates :**

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 \quad , \quad U = mgy$$

-Where U = 0 at y = 0, the lagrangian will be

$$L = T - U = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 - mgy$$

-We have two generalized coordinates (x, y) , the equations of motion are :

x:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$0 - \frac{d}{dt}m\dot{x} = 0$$

$$\ddot{x} = 0$$

y:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0$$

$$-mg - \frac{d}{dt}(m\dot{y}) = 0$$

$$\ddot{y} = -g$$

- **Second in Polar coordinates :**

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 \quad , \quad U = mgr\sin\theta$$

-Where U = 0 at y = 0, the lagrangian will be

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgr\sin\theta$$

-We have two generalized coordinates (r, θ) , the equations of motion are :

r:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$mr\dot{\theta}^2 - mg\sin\theta - \frac{d}{dt}(m\dot{r}) = 0$$

$$r\dot{\theta}^2 - g\sin\theta - \ddot{r} = 0$$

θ:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$-mgr\cos\theta - \frac{d}{dt}(mr^2\dot{\theta}) = 0$$

$$-gr\cos\theta - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0$$

The equations of motion expressed by (x, y) are clearly simpler than those of (r, θ)

Example 7.4 :

Consider the motion of a particle of mass *m that is constrained to move* on the surface of a cone of half-angle a and which is subject to a gravitational force.

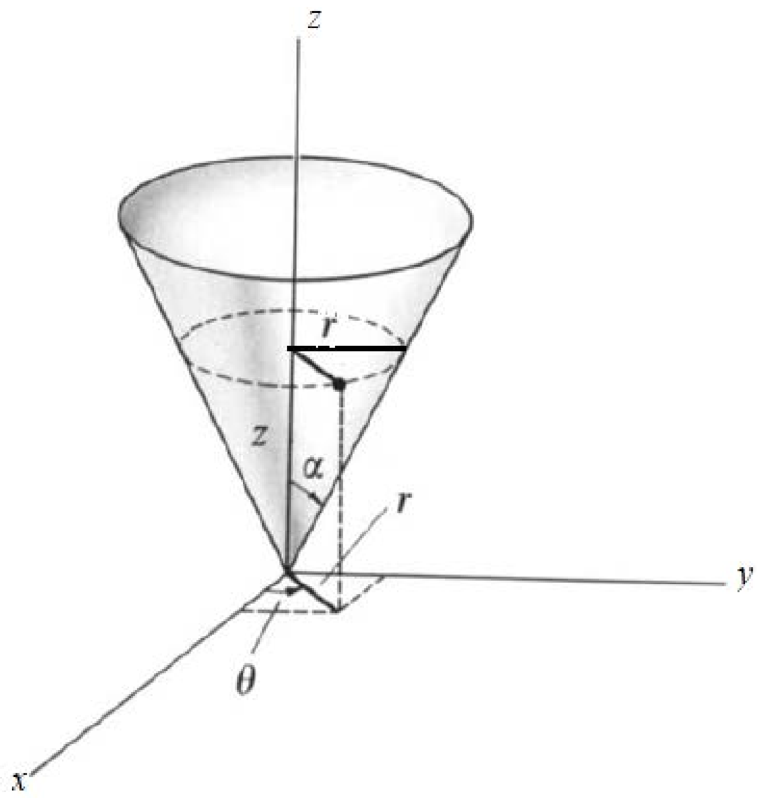
Solution :

Since the problem possesses cylindrical symmetry, we choose r, θ, and z as the generalized coordinates.

We have, however, the equation of constraint

$$z = r \cot \alpha$$

there are only two degrees of freedom for the system and therefore there are only two proper generalized coordinates



-the square of the velocity is

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2$$

$$= \dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2 \cos^2 \alpha$$

$$= \dot{r}^2 \csc^2 \alpha + r^2\dot{\theta}^2$$

-if we choose *U(z = 0) = 0 ,* The potential energy is

$$U = mgz = mgr \cot \alpha$$

-so that the Lagrangian is

$$L = \tfrac{1}{2}m(\dot{r}^2 \csc^2 \alpha + r^2\dot{\theta}^2) - mgr \cot \alpha$$

-We note first that *L does not explicitly contain θ.*

Therefore *dL/d θ = 0, and the* Lagrange equation for the coordinate *θ is*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Longrightarrow \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{const.}$$

This is, expresses the conservation of angular momentum about the axis of symmetry

The Lagrange equation for *r is*

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

And ,

$$L = \tfrac{1}{2}m(\dot{r}^2 \csc^2 \alpha + r^2\dot{\theta}^2) - mgr \cot \alpha$$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 - m g \cot\alpha \longrightarrow (1)$$

$$\frac{\partial L}{\partial \dot{r}} = m \dot{r} \csc^2\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m \ddot{r} \csc^2\alpha \longrightarrow (2)$$

-Use eq. (1) , (2) in Lagrange equation, we obtain :

- $\cancel{m} r \dot{\theta}^2 - \cancel{m} g \cot\alpha - \cancel{m} \ddot{r} \csc^2\alpha = 0$ and $\therefore \cot\alpha = \frac{\cos\alpha}{\sin\alpha}$, $\csc\alpha = \frac{1}{\sin\alpha}$

-Now mutiply by (- sin2α) and divide by (m) we obtain the equation of motion for the coordinate *r.* :

$$\ddot{r} - r\dot{\theta}^2 \sin^2\alpha + g \sin\alpha \cos\alpha = 0$$

Example 7.5 :

A point of support of a simple pendulum of length *b moves on a massless rim of* radius *a rotating with constant angular velocity ω. Obtain the expression for the Cartesian* components of the velocity and acceleration of the mass *m. Obtain also the angular* acceleration for angle *θ.*

Solution :

We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass ***m*** *become*

$$\left.\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}\right\}$$

And the velocities

$$\left.\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\dot{\theta}\cos\theta \\ \dot{y} &= a\omega\cos\omega t + b\dot{\theta}\sin\theta \end{aligned}\right\}$$

And the acceleration in Cartesian coordinates will be

$$\ddot{x} = -a\omega^2\cos\omega t + b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$$

$$\ddot{y} = -a\omega^2\sin\omega t + b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$$

$$\dot{x}^2 = (-a\omega \sin\omega t + b\dot{\theta}\cos\theta)^2$$

$$= a^2\omega^2\sin^2\omega t + b^2\dot{\theta}^2\cos^2\theta - 2ab\omega\dot{\theta}\sin\omega t\cos\theta \quad \rightarrow (1)$$

$$\dot{y}^2 = (a\omega\cos\omega t + b\dot{\theta}\sin\theta)^2$$

$$= a^2\omega^2\cos^2\omega t + b^2\dot{\theta}^2\sin^2\theta + 2a\omega b\dot{\theta}\sin\theta\cos\omega t \quad \rightarrow (2)$$

$$\Rightarrow \dot{x}^2 + \dot{y}^2 = a^2\omega^2(\sin^2\omega t + \cos^2\omega t) + b^2\dot{\theta}^2(\cos^2\theta + \sin^2\theta) + 2ab\omega\dot{\theta}(\sin\theta\cos\omega t - \cos\theta\sin\omega t)$$

$$\dot{x}^2 + \dot{y}^2 = a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\dot{\theta}\omega\sin(\theta - \omega t)$$

-It should be clear that the generalized coordinates is only θ , the kinetic energy and potential energy are

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$
$$U = mgy$$

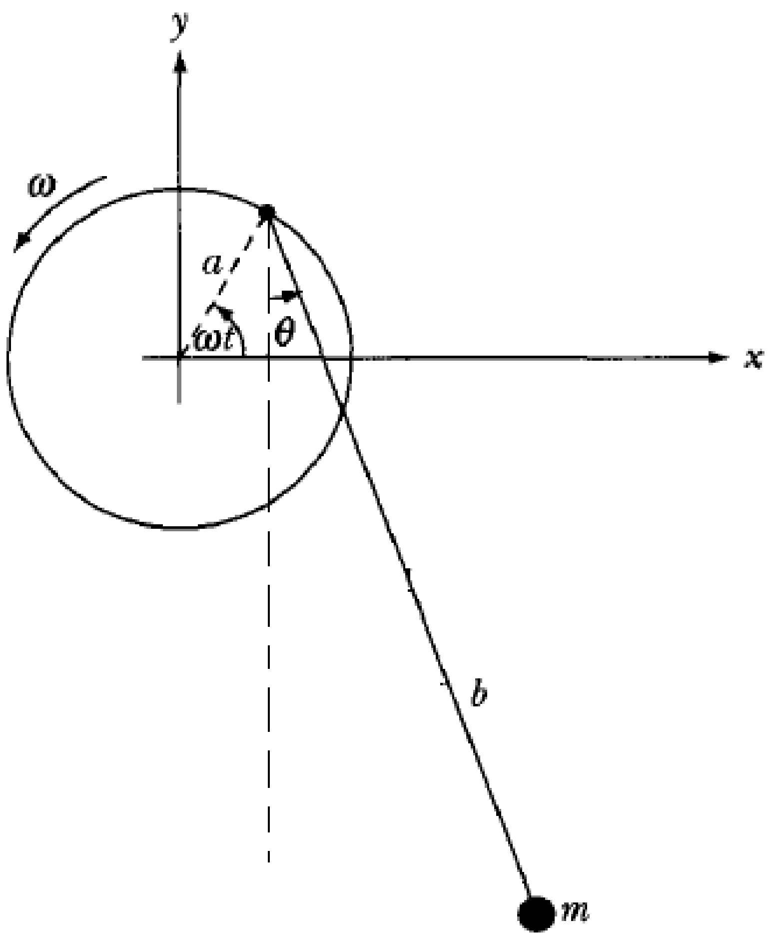
Where U = 0 at y = 0, the lagrangian will be

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega \sin(\theta - \omega t)] - mg(a \sin \omega t - b \cos \theta)$$

-So that,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = mb\dot{\theta}a\omega \cos(\theta - \omega t) - mgb \sin \theta$$



-Now applying Lagrange equation of motion,

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$\cancel{mb\dot{\theta}a\omega \cos(\theta-\omega t)} - mgb\sin\theta$$
$$- mb^2\ddot{\theta} - mb\cancel{a\omega\dot{\theta}\cos(\theta-\omega t)}$$
$$+ mba\omega^2\cos(\theta-\omega t) = 0$$

-Divide by (mb^2) and rearrange ;

$$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta$$

-**Example 7.6 :**

A bead slides along a smooth wire bent in the shape of a parabola $z = cr^2$ (Figure 7-5). The bead rotates in a circle of radius R when the wire is rotating about its vertical symmetry axis with angular velocity ω. Find the value of c.

-**Solution :**

Because the problem has cylindrical symmetry, we choose r, θ, and z as the generalized coordinates. The kinetic energy of the bead is

$$T = \frac{m}{2}[\dot{r}^2 + \dot{z}^2 + (r\dot{\theta}^2)]$$

If we choose $U = 0$ at $z = 0$, the potential energy term is

$$U = mgz$$

But r, z, and θ are not independent. The equation of constraint for the parabola is

$$z = cr^2$$

$$\dot{z} = 2c\dot{r}r$$

We also have an explicit time dependence of the angular rotation

$$\theta = \omega t$$

$$\dot{\theta} = \omega$$

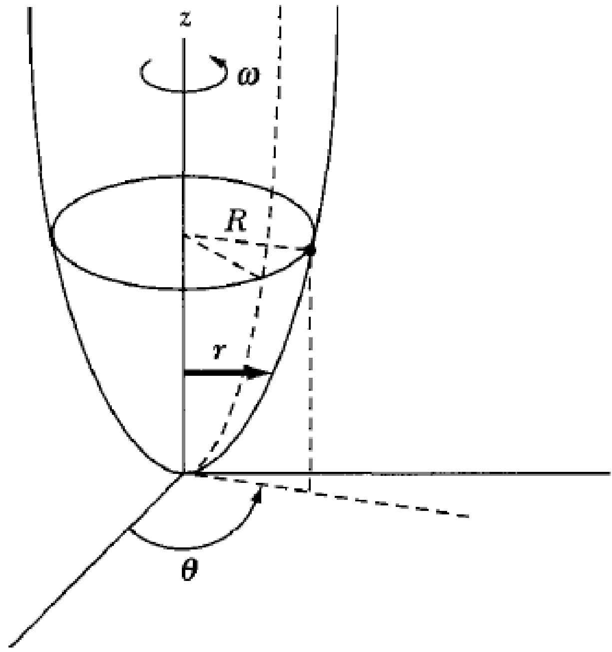
We can now construct the Lagrangian as being dependent only on r, because there is no direct θ dependence.

$$\begin{aligned} L &= T - U \\ &= \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2 \end{aligned}$$

$$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\dot{r} + 8c^2r^2\dot{r})$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}(2\ddot{r} + 16c^2r\dot{r}^2 + 8c^2r^2\ddot{r})$$

$$\frac{\partial L}{\partial r} = m(4c^2r\dot{r}^2 + r\omega^2 - 2gcr)$$



Lagrange's equation of motion becomes

$$\ddot{r}(1 + 4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc - \omega^2) = 0$$

which is a complicated result. If, however, the bead rotates with $r = R =$ constant, then $\dot{r} = \ddot{r} = 0$, and Equation 7.49 becomes

$$R(2gc - \omega^2) = 0$$

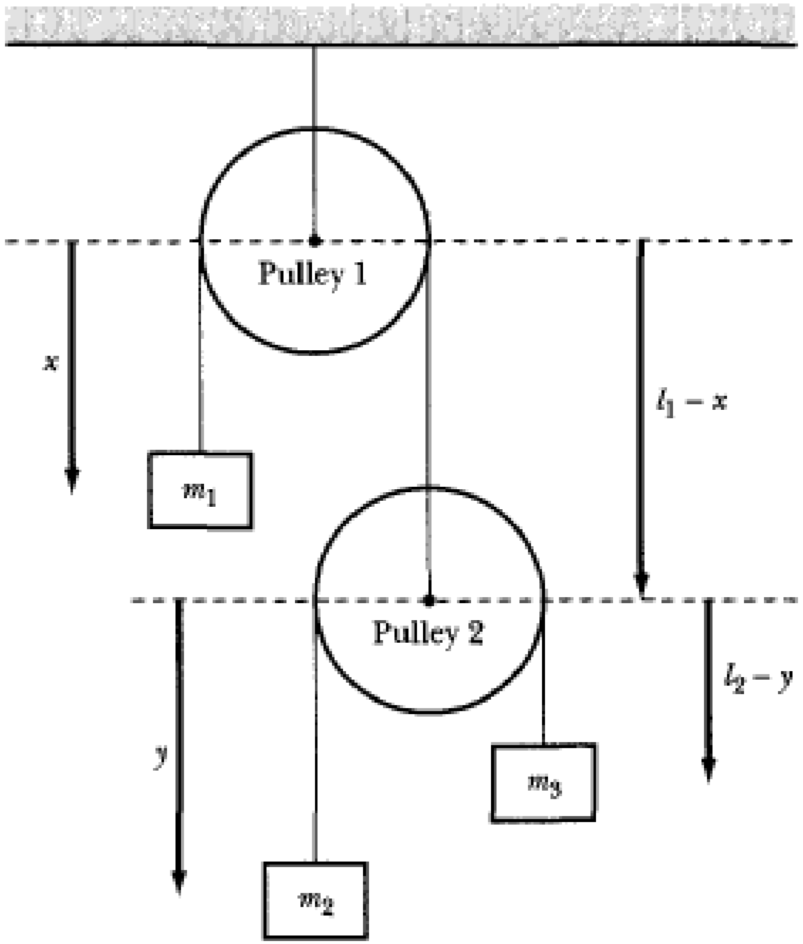
and

$$c = \frac{\omega^2}{2g}$$

-Example 7.8 :

Consider the double pulley system shown in Figure Use the coordinates indicated, and determine the equations of motion.

-Solution :



Consider the pulleys to be massless, and let l_1 and l_2 be the lengths of rope hanging freely from each of the two pulleys. The distances x and y are measured from the center of the two pulleys.

m_1:

$$v_1 = \dot{x}$$

m_2:

$$v_2 = \frac{d}{dt}(l_1 - x + y) = -\dot{x} + \dot{y}$$

m_3:

$$v_3 = \frac{d}{dt}(l_1 - x + l_2 - y) = -\dot{x} - \dot{y}$$

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{y} - \dot{x})^2 + \frac{1}{2}m_3(-\dot{x} - \dot{y})^2$$

Let the potential energy $U = 0$ at $x = 0$.

$$U = U_1 + U_2 + U_3$$

$$= -m_1 gx - m_2 g(l_1 - x + y) - m_3 g(l_1 - x + l_2 - y)$$

Because T and U have been determined, the equations of motion can be obtained using Equation 7.18. The results are

$$m_1\ddot{x} + m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_1 - m_2 - m_3)g$$

$$-m_2(\ddot{x} - \ddot{y}) + m_3(\ddot{x} + \ddot{y}) = (m_2 - m_3)g$$

Equations 7.56 and 7.57 can be solved for $\ddot{x}$ and $\ddot{y}$.

7.5 Lagrange's Equations with Undetermined Multipliers

If the constraint relations for a problem are given in differential form rather than as algebraic expressions, we can incorporate them directly into Lagrange's equations by using the Lagrange undetermined multipliers ; that is, for constraints expressible as

$$\sum_j \frac{\partial f_k}{\partial q_j} dq_j = 0 \quad \begin{cases} j = 1, 2, \ldots, s \\ k = 1, 2, \ldots, m \end{cases}$$

the Lagrange equations are

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_k \lambda_k(t) \frac{\partial f_k}{\partial q_j} = 0$$

the undetermined multipliers $\lambda_k(t)$ are closely related to the forces of constraint.

The generalized forces of constraint Q_j are given by

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j}$$

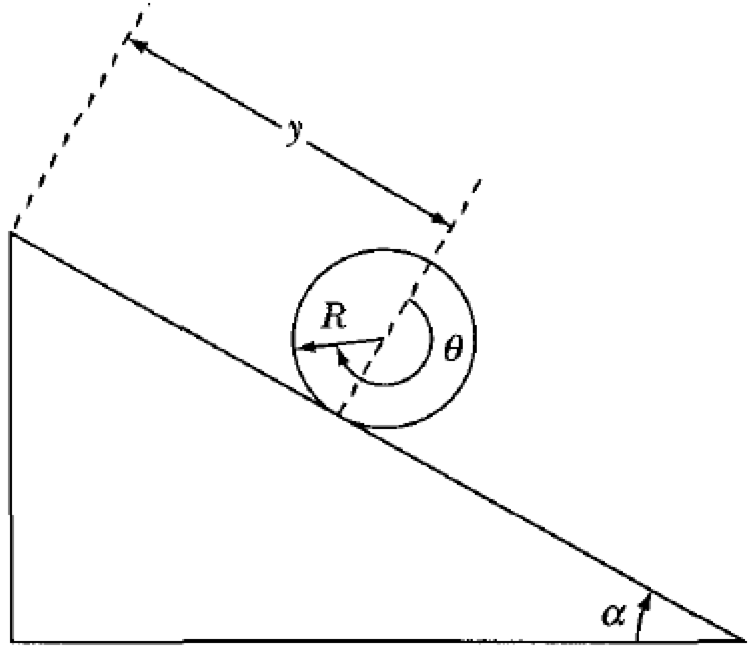
-Example 7.9 :

Let us consider again the case of the disk rolling down an inclined plane (see Example 6.5 and Figure 6-7). Find the equations of motion, the force of constraint, and the angular acceleration.

-Solution :

The equation of constraint is

$$f(y, \theta) = y - R\theta = 0 \quad \text{———} \quad (1)$$



The kinetic energy may be separated into translational and rotational terms

$$T = \frac{1}{2} M\dot{y}^2 + \frac{1}{2} I\dot{\theta}^2$$

$$= \frac{1}{2} M\dot{y}^2 + \frac{1}{4} MR^2 \dot{\theta}^2$$

where M is the mass of the disk and R is the radius; $I = \frac{1}{2} MR^2$ is the moment of inertia of the disk about a central axis. The potential energy is

$$U = Mg(l - y) \sin \alpha$$

where l is the length of the inclined surface of the plane and where the disk is assumed to have zero potential energy at the bottom of the plane. The Lagrangian is therefore

$$L = T - U$$

$$= \frac{1}{2}M\dot{y}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mg(y - l) \sin \alpha$$

The system has only one degree of freedom if we insist that the rolling takes place without slipping. We may therefore choose either y or θ as the proper coordinate and use Equation (1) to eliminate the other. Alternatively, we may continue to consider *both* y and θ as generalized coordinates and use the method of undetermined multipliers. The Lagrange equations in this case are

$$\left.\begin{aligned} \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda\frac{\partial f}{\partial y} = 0 \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \end{aligned}\right\} \qquad \textbf{(7.70)}$$

Performing the differentiations, we obtain, for the equations of motion,

$$Mg\sin\alpha - M\ddot{y} + \lambda = 0 \qquad \textbf{(7.71a)}$$

$$-\frac{1}{2}MR^2\ddot{\theta} - \lambda R = 0 \qquad \textbf{(7.71b)}$$

Also, from the constraint equation, we have

$$y = R\theta \qquad \textbf{(7.72)}$$

These equations (Equations 7.71 and 7.72) constitute a soluble system for the three unknowns y, θ, λ. Differentiating the equation of constraint (Equation 7.72), we obtain

$$\ddot{\theta} = \frac{\ddot{y}}{R} \tag{7.73}$$

Combining Equations 7.71b and 7.73, we find

$$\lambda = -\frac{1}{2}M\ddot{y} \tag{7.74}$$

and then using this expression in Equation 7.71a there results

$$\ddot{y} = \frac{2g\sin\alpha}{3} \tag{7.75}$$

with

$$\lambda = -\frac{Mg\sin\alpha}{3} \tag{7.76}$$

so that Equation 7.71b yields

$$\ddot{\theta} = \frac{2g\sin\alpha}{3R} \tag{7.77}$$

Thus, we have three equations for the quantities $\ddot{y}$, $\ddot{\theta}$, and λ that can be immediately integrated.

We note that if the disk were to slide without friction down the plane, we would have $\ddot{y} = g \sin \alpha$. Therefore, the rolling constraint reduces the acceleration to $\frac{2}{3}$ of the value of frictionless sliding. The magnitude of the force of friction producing the constraint is just λ—that is, $(Mg/3) \sin \alpha$.

The generalized forces of constraint, Equation 7.66, are

$$Q_y = \lambda \frac{\partial f}{\partial y} = \lambda = -\frac{Mg \sin \alpha}{3}$$

$$Q_\theta = \lambda \frac{\partial f}{\partial \theta} = -\lambda R = \frac{MgR \sin \alpha}{3}$$

Note that Q_y and Q_θ are a force and a torque, respectively, and they are the generalized forces of constraint required to keep the disk rolling down the plane without slipping.

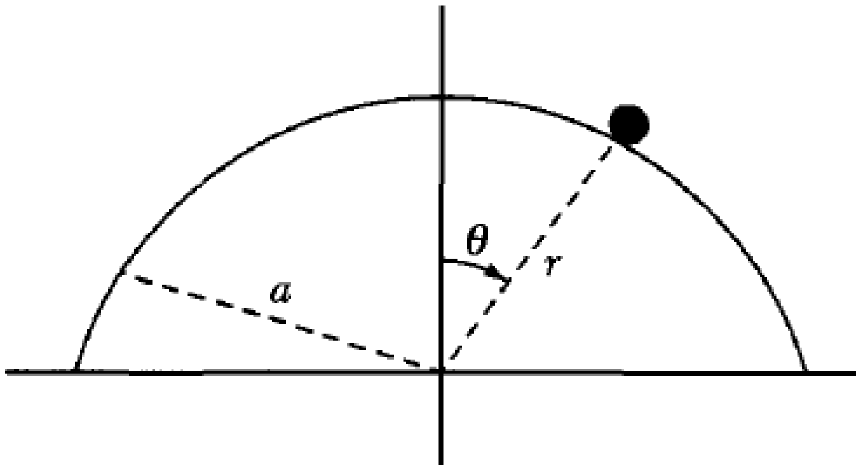
Example 7.10 :

A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius a. Find the force of constraint, and determine the angle at which the particle leaves the hemisphere.

-Solution :

Because we are considering the possibility of the particle leaving the hemisphere, we choose the generalized coordinates to be r and θ. The constraint equation is

$$f(r, \theta) = r - a = 0 \qquad \textbf{(7.80)}$$



The Lagrangian is determined from the kinetic and potential energies:

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

$$U = mgr \cos \theta$$

$$L = T - U$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - mgr \cos \theta \qquad \textbf{(7.81)}$$

where the potential energy is zero at the bottom of the hemisphere. The Lagrange equations, Equation 7.65, are

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \tag{7.82}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \tag{7.83}$$

Performing the differentiations on Equation 7.80 gives

$$\frac{\partial f}{\partial r} = 1, \qquad \frac{\partial f}{\partial \theta} = 0 \tag{7.84}$$

Equations 7.82 and 7.83 become

$$mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0 \tag{7.85}$$

$$mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} = 0 \tag{7.86}$$

Next, we apply the constraint $r = a$ to these equations of motion:

$$r = a, \quad \dot{r} = 0 = \ddot{r}$$

Equations 7.85 and 7.86 then become

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0 \tag{7.87}$$

$$mga\sin\theta - ma^2\ddot{\theta} = 0 \tag{7.88}$$

From Equation 7.88, we have

$$\ddot{\theta} = \frac{g}{a}\sin\theta \tag{7.89}$$

We can integrate Equation 7.89 to determine $\dot{\theta}^2$.

$$\ddot{\theta} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} \tag{7.90}$$

We integrate Equation 7.89,

$$\int\dot{\theta}\,d\dot{\theta} = \frac{g}{a}\int\sin\theta\,d\theta \tag{7.91}$$

which results in

$$\frac{\dot{\theta}^2}{2} = \frac{-g}{a}\cos\theta + \frac{g}{a} \tag{7.92}$$

where the integration constant is g/a, because $\dot{\theta} = 0$ at $t = 0$ when $\theta = 0$. Substituting $\dot{\theta}^2$ from Equation 7.92 into Equation 7.87 gives, after solving for λ,

$$\lambda = mg(3\cos\theta - 2) \tag{7.93}$$

which is the force of constraint. The particle falls off the hemisphere at angle θ_0 when $\lambda = 0$.

$$\lambda = 0 = mg(3\cos\theta_0 - 2) \tag{7.94}$$

$$\theta_0 = \cos^{-1}\left(\frac{2}{3}\right) \tag{7.95}$$

As a quick check, notice that the constraint force is $\lambda = mg$ at $\theta = 0$ when the particle is perched on top of the hemisphere.

7.9 Conservation theorms

The Conservation of Energy:

According to our previous arguments, time is homogeneous within an inertial reference frame. Therefore, the Lagrangian that describes a *closed system (i.e., a system which does not interact with anything outside* the system) cannot depend explicitly on the time

$$\frac{\partial L}{\partial t} = 0 \tag{7.124}$$

so that the total derivative of the Lagrangian becomes

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j \tag{7.125}$$

where the usual term, $\partial L/\partial t$, does not now appear. But Lagrange's equations are

$$\frac{\partial L}{\partial q_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \tag{7.126}$$

Using Equation 7.126 to substitute for $\partial L/\partial q_j$ in Equation 7.125, we have

$$\frac{dL}{dt} = \sum_j \dot{q}_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \qquad \text{or} \qquad \frac{dL}{dt} - \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

so that

$$\frac{d}{dt}\left(L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) = 0 \tag{7.127}$$

The quantity in the parentheses is therefore constant in time; denote this constant by $-H$:

$$L - \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} = -H = \text{constant} \tag{7.128}$$

where we exclude the possibility of an explicit time dependence in the transformation equations. Therefore, $U = U(q_j)$, and $\partial U/\partial \dot{q}_j = 0$. Thus

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial (T - U)}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$$

Equation 7.128 can then be written as

$$(T - U) - \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = -H \tag{7.129}$$

and, using Equation 7.122, we have

$$(T - U) - 2T = -H \qquad \text{or} \qquad T + U = E = H = \text{constant} \tag{7.130}$$

The total energy E is a constant of the motion for this case.

The function H, called the **Hamiltonian** of the system, may be defined as in Equation 7.128 (but see Section 7.10). It is important to note that the Hamiltonian H is equal to the total energy E only if the following conditions are met:

1. The equations of the transformation connecting the rectangular and generalized coordinates (Equation 7.116) must be independent of the time, thus ensuring that the kinetic energy is a homogeneous quadratic function of the $\dot{q}_j$.
2. The potential energy must be velocity independent, thus allowing the elimination of the terms $\partial U/\partial \dot{q}_i$ from the equation for H (Equation 7.129).

The Conservation of Linear Momentum:

Since space is homogeneous in an inertial reference frame, the Lagrangian of a closed system will be unaffected by a translation of the entire system in space. Consider an infinitesimal translation of every radius vector ra such that ra $\Longrightarrow$ ra + δr; this amounts to translating the entire system by δ*r*.

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i + \sum_i \frac{\partial L}{\partial \dot{x}_i}\delta \dot{x}_i = 0$$

Now, we consider only a *displacement*, so that the δx_i are not functions of the time. Thus,

$$\delta \dot{x}_i = \delta \frac{dx_i}{dt} = \frac{d}{dt}\delta x_i \equiv 0$$

Therefore, δL becomes

$$\delta L = \sum_i \frac{\partial L}{\partial x_i}\delta x_i = 0$$

Since each of the δx_i is an independent displacement, δL will vanish identically only if each of the partial derivatives of L vanishes:

$$\frac{\partial L}{\partial x_i} = 0$$

Then, according to Lagrange's equations, $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}_i} = 0$

and,

$$\frac{\partial L}{\partial \dot{x}_i} = \text{const.}$$

or,

$$\frac{\partial (T - U)}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left(\tfrac{1}{2}m \sum_j \dot{x}_j^2\right)$$

$$= m\dot{x}_i = p_i = \text{const.}$$

Thus, the homogeneity of space implies that the linear momentum **p** of a closed system is constant in time.

This result may also be interpreted according to the following statement: If the Lagrangian of a system (not necessarily *closed*) is invariant with respect to translation in a certain direction, then the linear momentum of the system in that direction is constant in time.

The Conservation of Angular Momentum:

one characteristic of an inertial reference frame is that space is isotropie; i.e., the mechanical properties of a closed system are unaffected by the orientation of the system. In particular, the Lagrangian of a closed system will not change if the system is rotated

through an infinitesimal angle

If a system is rotated about a certain axis by an infinitesimal angle $\delta\theta$, Since ($\delta\theta$ is arbitrary, we must have

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = 0$$

and,

$$\mathbf{r} \times \mathbf{p} = \text{const.}$$

But $\mathbf{r} \times \mathbf{p} = \mathbf{L}$; therefore, the angular momentum of the particle is constant in time.

TABLE 7-1

Characteristic of inertial frame	Property of Lagrangian	Conserved quantity
Time homogeneous	Not explicit function of time	Total energy
Space homogeneous	Invariant to translation	Linear momentum
Space isotropic	Invariant to rotation	Angular momentum

7.10 The Canonical Equations of Motion—Hamiltonian Dynamics

the Lagrangian is expressed in generalized coordinates and define the *generalized momenta* according to

$$\boxed{p_j \equiv \frac{\partial L}{\partial \dot{q}_j}}$$

The Lagrange equations of motion are then expressed by

$$\boxed{\dot{p}_j = \frac{\partial L}{\partial q_j}}$$

Using the definition of the generalized momenta for the Hamiltonian may be written as

$$\boxed{H(q_k, p_k, t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k, t)}$$

This equation is written in a manner which stresses the fact that the Hamiltonian is always considered as a function of $H = H(q_k, p_k, t)$; $L = L(q_k, \dot{q}_k, t)$

By using the previous equation we find

$$\boxed{\begin{aligned} \dot{q}_k &= \frac{\partial H}{\partial p_k} \\ -\dot{p}_k &= \frac{\partial H}{\partial q_k} \end{aligned}}$$

Hamilton's equations of motion

and

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Hamilton's equations of motion; because of their symmetrical appearance, they are also known as the *canonical equations of motion. The description of motion by means of these* equations is termed *Hamiltonian dynamics.*

The last equation expresses the fact that if *H does not explicitly contain* the time, then the Hamiltonian is a conserved quantity.

There are *2s canonical equations and they replace the s Lagrange* equations. (Recall that *$s = 3n - m$ is the number of degrees of freedom of* the system.) But the canonical equations are *first-order differential equations,* whereas the Lagrange equations are of *second-order.*

In order to use the canonical equations in solving a problem, the Hamiltonian must first be constructed as a function of the generalized coordinates and momenta.

EXAMPLE 7.11

Use the Hamiltonian method to find the equations of motion of a particle of mass m constrained to move on the surface of a cylinder defined by . $x^2 + y^2 = R^2$. The particle is subject to a force directed toward the origin and proportional to the distance of the particle from the origin: $\mathbf{F} = -k\mathbf{r}$.

Solution. The situation is illustrated in Figure 7-9. The potential corresponding to the force **F** is

$$\begin{aligned} U &= \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2) \\ &= \frac{1}{2}k(R^2 + z^2) \end{aligned} \tag{7.164}$$

We can write the square of the velocity in cylindrical coordinates (see Equation 1.101) as

$$v^2 = \dot{R}^2 + \dot{R}^2\dot{\theta}^2 + \dot{z}^2 \tag{7.165}$$

But in this case, R is a constant, so the kinetic energy is

$$T = \frac{1}{2}m(R^2\dot{\theta}^2 + \dot{z}^2) \tag{7.166}$$

We may now write the Lagrangian as

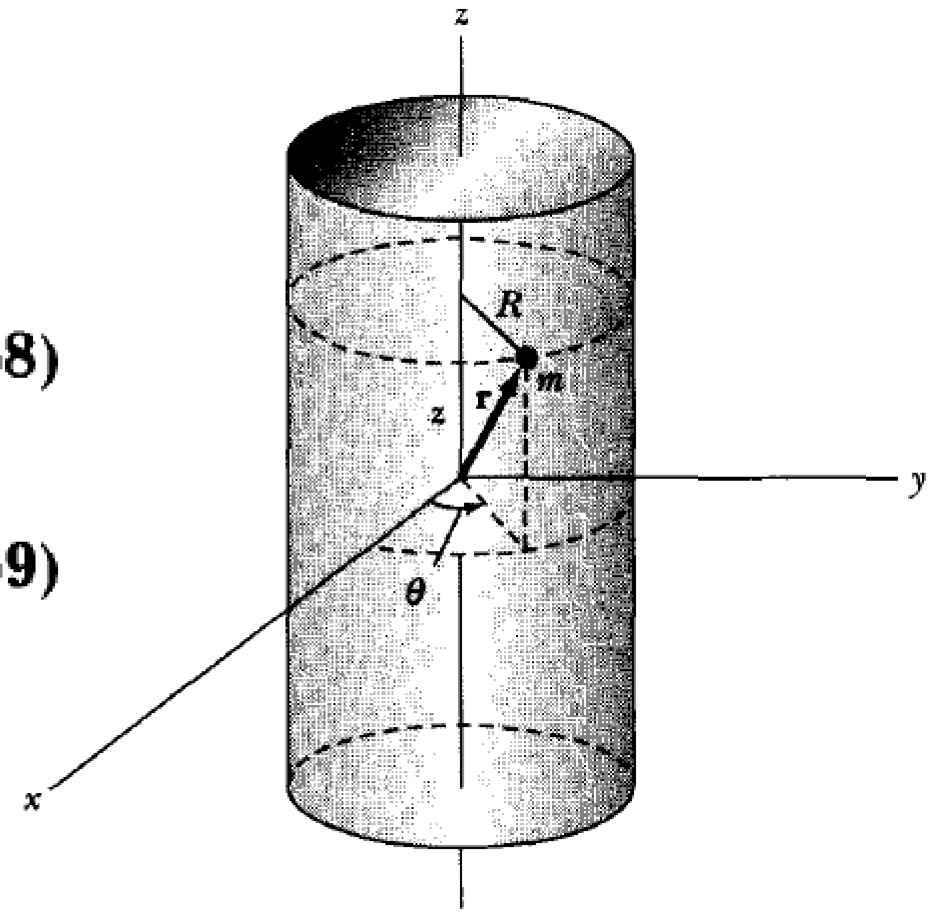
$$L = T - U = \frac{1}{2} m(R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2} k(R^2 + z^2) \quad \textbf{(7.167)}$$

The generalized coordinates are θ and z,

and the generalized momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad \textbf{(7.168)}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \textbf{(7.169)}$$



Example 7.11. A particle is constrained to move on the surface of a cylinder.

We may now write the Lagrangian as

$$L = T - U = \frac{1}{2} m(R^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2} k(R^2 + z^2) \tag{7.167}$$

The generalized coordinates are θ and z, and the generalized momenta are

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \tag{7.168}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \tag{7.169}$$

Because the system is conservative and because the equations of transformation between rectangular and cylindrical coordinates do not explicitly involve the time, the Hamiltonian H is just the total energy expressed in terms of the variables θ, p_θ, z, and p_z. But θ does not occur explicitly, so

$$\begin{aligned} H(z, p_\theta, p_z) &= T + U \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2} kz^2 \end{aligned} \tag{7.170}$$

where the constant term $\frac{1}{2}kR^2$ has been suppressed. The equations of motion are therefore found from the canonical equations:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{7.171}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz \tag{7.172}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \tag{7.173}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \tag{7.174}$$

Equations 7.173 and 1.174 just duplicate Equations 7.168 and 7.169. Equations 7.168 and 7.171 give

$$p_\theta = mR^2\dot{\theta} = \text{constant} \tag{7.175}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \tag{7.173}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \tag{7.174}$$

Equations 7.173 and 1.174 just duplicate Equations 7.168 and 7.169. Equations 7.168 and 7.171 give

$$p_\theta = mR^2\dot{\theta} = \text{constant} \tag{7.175}$$

EXAMPLE 7.12

Use the Hamiltonian method to find the equations of motion for a spherical pendulum of mass m and length b (see Figure 7-10).

Solution. The generalized coordinates are θ and ϕ. The kinetic energy is

$$T = \frac{1}{2} mb^2\dot{\theta}^2 + \frac{1}{2} mb^2 \sin^2 \theta\dot{\phi}^2$$

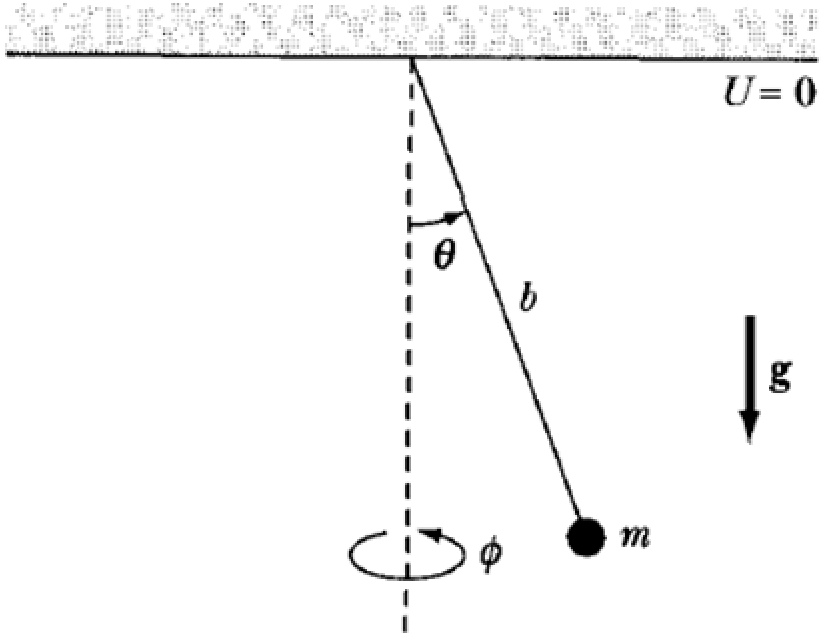
The only force acting on the pendulum (other than at the point of support) is gravity, and we define the potential zero to be at the pendulum's point of attachment.

$$U = -mgb \cos \theta$$

The generalized momenta are then

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta} \tag{7.180}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mb^2 \sin^2 \theta\dot{\phi} \tag{7.181}$$



Example 7.12. A spherical pendulum with generalized coordinates θ and ϕ.

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2}mb^2\frac{p_\theta^2}{(mb^2)^2} + \frac{1}{2}\frac{mb^2 \sin^2 \theta p_\phi^2}{(mb^2 \sin^2 \theta)^2} - mgb \cos \theta \\ &= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2 \sin^2 \theta} - mgb \cos \theta \end{aligned}$$

The equations of motion are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2 \sin^2 \theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2 \cos \theta}{mb^2 \sin^3 \theta} - mgb \sin \theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

Because ϕ is cyclic, the momentum p_ϕ about the symmetry axis is constant.