Some Methods in the Calculus of Variations

Ch. 6

6.1 Introduction

The development of the calculus of variations was begun by Newton (1686) and was extended by Johann and Jakob Bernoulli (1696) and by Euler (1744). Adrien Legendre (1786), Joseph Lagrange (1788), Hamilton (1833), and Jacobi (1837) all made important contributions. The names of Peter Dirichlet (1805–1859) and Karl Weierstrass (1815–1879) are particularly associated with the establishment of a rigorous mathematical foundation for the subject.

Many problems in Newtonian mechanics are more easily analyzed by means of alternative statements of the laws, including **Lagrange's equation** and **Hamilton's principle.*** As a prelude to these techniques, we consider in this chapter some general principles of the techniques of the calculus of variations.

Our primary interest here is in determining the path that gives extremum solutions, for example, the shortest distance (or time) between two points.

6.2 Statement of the problem

-- The basic problem of the calculus of variations is to determine the function $y(x)$

such that the integral $$J = \int_{x_1}^{x_2} f\{y(x), y'(x); x\}\, dx$$ is an *extremum*

-- if a function $y = y(x)$ gives the integral J a minimum value,

then any *neighboring function*, no matter how close to $y(x)$, must make J increase.

-- The definition of a neighboring function may be made as follows.

We give all possible functions y a parametric representation $y = y(\alpha, x)$ such that,

for $\alpha = 0$, $y = y(0, x) = y(x)$ is the function that yields an extremum for J.

We can then write $$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

where $\eta(x)$ is some function of x and $\eta(x_1) = \eta(x_2) = 0$.

the integral J becomes a functional of the parameter α:

$$J(\alpha) = \int_{x_1}^{x_2} f\{y(\alpha, x), y'(\alpha, x); x\}\, dx$$

The condition that the integral is an extremum

is $$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0 \qquad \text{for all functions } \eta(x).$$

EXAMPLE 6.1

Consider the function $f = (dy/dx)^2$, where $y(x) = x$. Add to $y(x)$ the function $\eta(x) = \sin x$, and find $J(\alpha)$ between the limits of $x = 0$ and $x = 2\pi$. Show that the stationary value of $J(\alpha)$ occurs for $\alpha = 0$.

Solution. We may construct neighboring varied paths by adding to $y(x)$,

$$y(x) = x \tag{6.5}$$

the sinusoidal variation $\alpha \sin x$,

$$y(\alpha, x) = x + \alpha \sin x \tag{6.6}$$

These paths are illustrated in Figure 6-2 for $\alpha = 0$ and for two different nonvanishing values of α. Clearly, the function $\eta(x) = \sin x$ obeys the endpoint conditions, that is, $\eta(0) = 0 = \eta(2\pi)$. To determine $f(y, y'; x)$ we first determine

$$\frac{dy(\alpha, x)}{dx} = 1 + \alpha \cos x \tag{6.7}$$

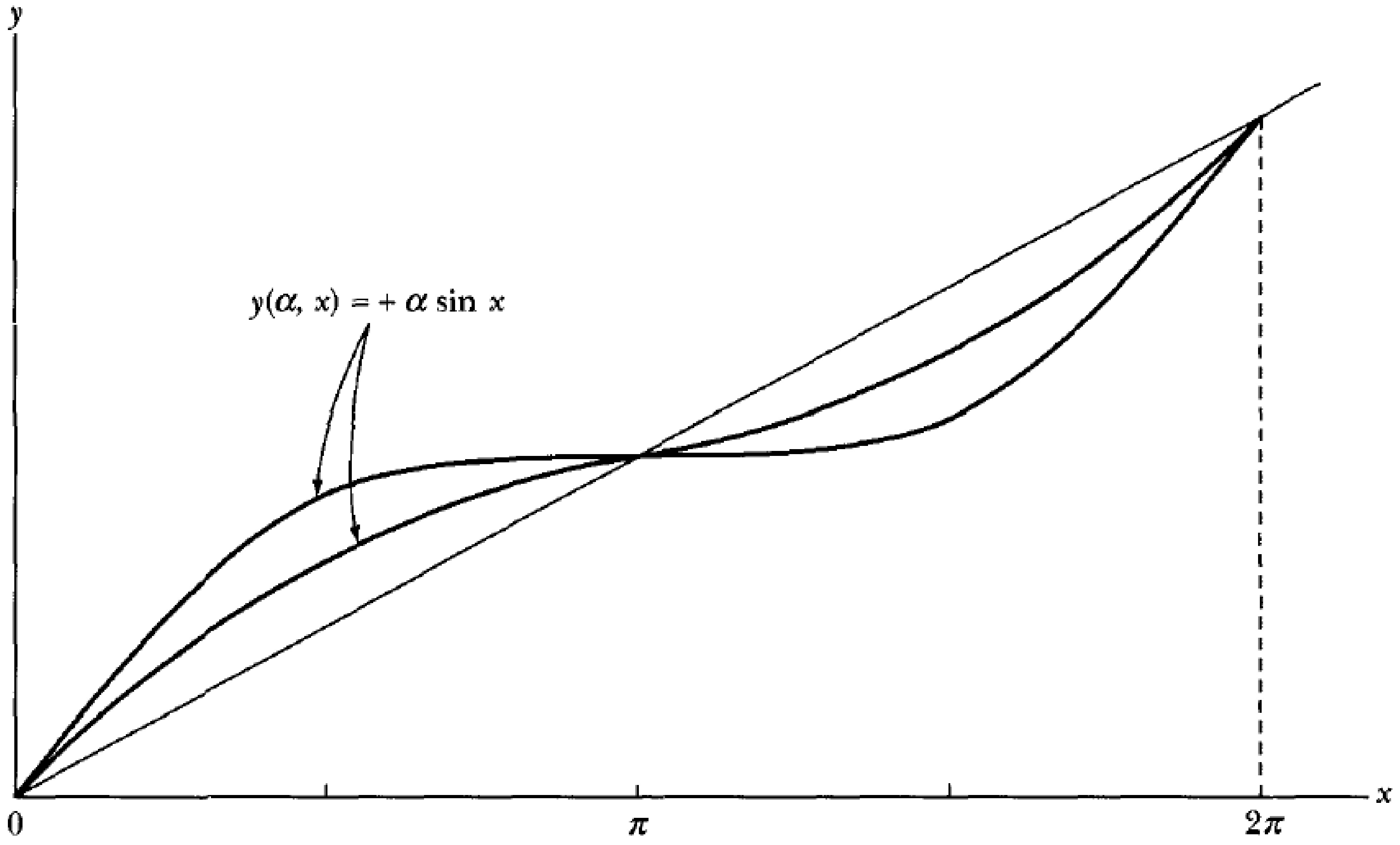


FIGURE 6-2 Example 6.1. The various paths $y(\alpha, x) = x + \alpha \sin x$. The extremum path occurs for $\alpha = 0$.

then

$$f = \left(\frac{dy(\alpha, x)}{dx}\right)^2 = 1 + 2\alpha \cos x + \alpha^2 \cos^2 x \tag{6.8}$$

Equation 6.3 now becomes

$$J(\alpha) = \int_0^{2\pi} (1 + 2\alpha \cos x + \alpha^2 \cos^2 x)\, dx \tag{6.9}$$

$$= 2\pi + \alpha^2 \pi \tag{6.10}$$

Thus we see the value of $J(\alpha)$ is always greater than $J(0)$, no matter what value (positive or negative) we choose for α. The condition of Equation 6.4 is also satisfied.

$$\int \cos^2 x \, dx$$

$$= \int \frac{\cos 2x}{2} + \frac{1}{2} \, dx$$

$$= \frac{\sin 2x}{2} \times \frac{1}{2} + \frac{x}{2} + c$$

$$= \frac{\sin 2x}{4} + \frac{x}{2} + c$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos 2x = \cos^2 x - (1 - \cos^2 x)$$

$$\cos 2x = \cos^2 x - 1 + \cos^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

$$2\cos^2 x = \cos 2x + 1$$

$$\cos^2 x = \frac{\cos 2x}{2} + \frac{1}{2}$$

6.3 Euler's Equation

$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\{y, y'; x\}\, dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx$$

$$\frac{\partial y}{\partial \alpha} = \eta(x); \qquad \frac{\partial y'}{\partial \alpha} = \frac{d\eta}{dx}$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx}\right) dx$$

The second term in the integrand can be integrated by parts:

$$\int u\,dv = uv - \int v\,du$$

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d\eta}{dx}\,dx = \frac{\partial f}{\partial y'}\eta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\,dx$$

because $\eta(x_1) = \eta(x_2) = 0$. Therefore,

$$\Rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\eta(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta(x)\right]dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\eta(x)\,dx$$

Because $(\partial J/\partial\alpha)|_{\alpha=0}$ must vanish for the extremum value and

because $\eta(x)$ is an arbitrary function, the integrand

must itself vanish for $\alpha = 0$:

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \qquad \text{Euler's equation}$$

Example 6.2 :

Consider a particle moving in a constant force field starting at rest from some point (x_1, y_1) to some lower point (x_2, y_2). Find the path that allows the particle to accomplish the transit in the least possible time.

Solution :

- According to the fig. We choose (x1, y1) at origin

- Constant field force without friction $\longrightarrow$ conservative force

$\longrightarrow$ T + U = 0 if the particle starts from rest with (U(0)=0)

let the force field be directed along the positive x-axis

$$T = \frac{1}{2}mv^2, \text{ and } \quad U = -Fx = -mgx, \quad \longrightarrow \quad v = \sqrt{2gx}$$

The time required for the particle to make the transit from the origin to (x_2, y_2) is

$$t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{ds}{v} = \int \frac{(dx^2 + dy^2)^{1/2}}{(2gx)^{1/2}}$$

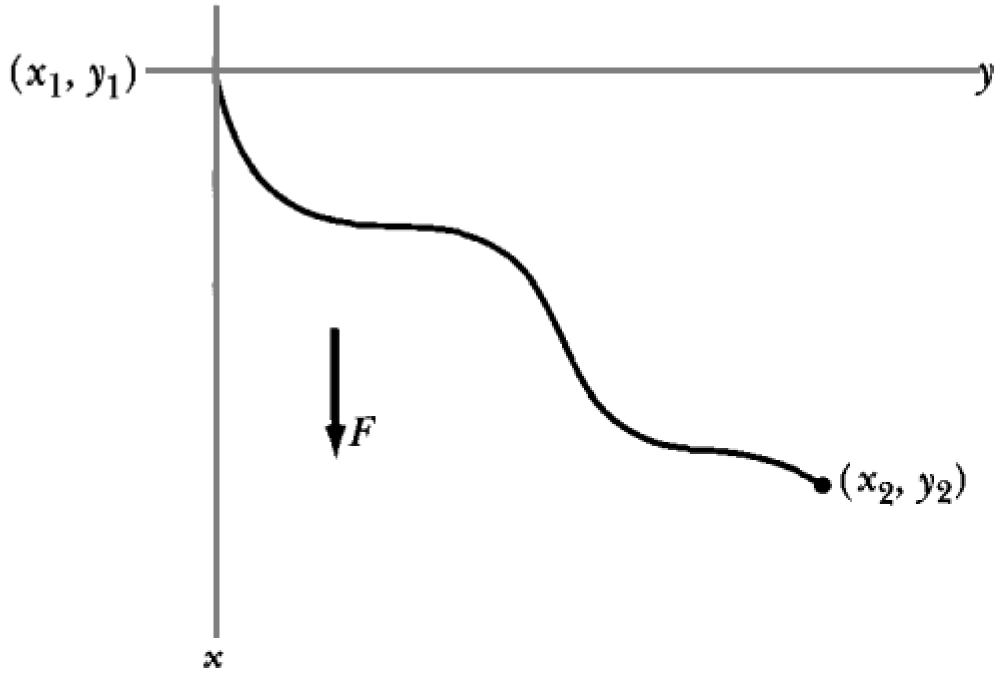
$$= \int_{x_1=0}^{x_2} \left(\frac{1 + y'^2}{2gx}\right)^{1/2} dx$$

-The unction ***f*** identified as :

$$f = \left(\frac{1 + y'^2}{x}\right)^{1/2} \longrightarrow (1)$$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

Euler's equation



the path of a particle moving from (x_1, y_1) to (x_2, y_2) that occurs in the least possible time. The force field acting on the particle is F, which is down and constant.

because $\partial f/\partial y = 0$, the Euler equation becomes

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad \longrightarrow \quad \frac{\partial f}{\partial y'} = \text{constant} \equiv (2a)^{-1/2}$$

where a is a new constant.

Differentiation eq (1) and squaring the result , we have :

$$\frac{y'^2}{x(1 + y'^2)} = \frac{1}{2a} \quad \longrightarrow \quad y = \int \frac{x\,dx}{(2ax - x^2)^{1/2}}$$

And by changing the variables : $x = a(1 - \cos\theta)$, $dx = a\sin\theta\,d\theta$

$$\longrightarrow \quad y = \int a(1 - \cos\theta)\,d\theta$$

$$y = a(\theta - \sin\theta) + \text{constant}$$

-If we take the constant = 0 then, the equation for y and x becomes :

$$\left.\begin{aligned} x &= a(1 - \cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}\right\}$$

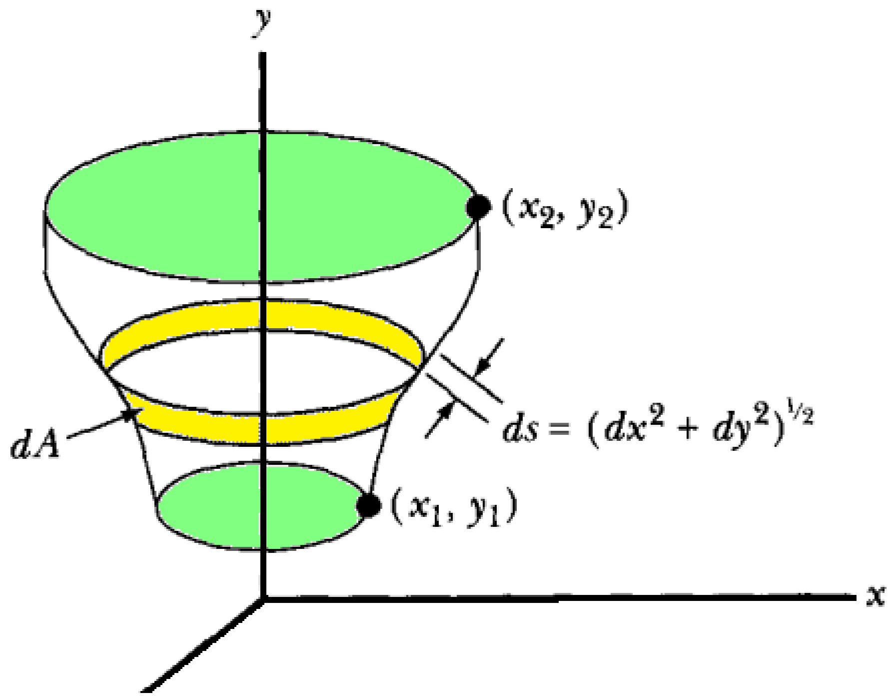
-Which are equations for cycloid passing through the orinign

Example 6.2 :

Consider the surface generated by revolving a line connecting two fixed points (x_1, y_1) and (x_2, y_2) about an axis coplanar with the two points. Find the equation of the line connecting the points such that the surface area generated by the revolution (i.e., the area of the surface of revolution) is a minimum.

Solution :

- Let the curve connecting two points revolved about y-axis
- Fist we find the area of a strip dA and then integrate to find the A



The geometry of the problem and area dA are indicated to minimize the surface of revolution around the y-axis.

-The area of the strip is:

$$dA = 2\pi x\, ds = 2\pi x(dx^2 + dy^2)^{1/2}$$

-And the total side area is :

$$A = 2\pi \int_{x_1}^{x_2} x(1 + y'^2)^{1/2}\, dx$$

-Let f be :

$$f = x(1 + y'^2)^{1/2}$$

-Now applying Euler equation :

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \qquad \text{Euler's equation}$$

- so that:

$$\frac{\partial f}{\partial y} = 0 \quad , \quad \frac{\partial f}{\partial y'} = \frac{xy'}{(1 + y'^2)^{1/2}}$$

$$\rightarrow \quad \frac{d}{dx}\left[\frac{xy'}{(1 + y'^2)^{1/2}}\right] = 0$$

$$\rightarrow \quad \frac{xy'}{(1 + y'^2)^{1/2}} = \text{constant} \equiv a$$

$$\rightarrow \quad y' = \frac{a}{(x^2 - a^2)^{1/2}}$$

or

$$y = \int \frac{a\,dx}{(x^2 - a^2)^{1/2}}$$

-The solution for this integral is

$$y = a \cosh^{-1}\left(\frac{x}{a}\right) + b$$

This is an equation of the curve of a flexible cord hanging freely between two points .

$$T + U = 0$$

$$\frac{1}{2}mv^2 + (-mgx) = 0$$

$$v^2 = 2gx$$

$$v = \sqrt{2gx}$$

$$f = \left(\frac{1+y'^2}{x}\right)^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{\cancel{2}}\left(\frac{1+y'^2}{x}\right)^{-\frac{1}{2}} \cdot \frac{\cancel{2}y'}{x}$$

$$= \frac{y'}{x\left(\frac{1+y'^2}{x}\right)^{\frac{1}{2}}} = (2a)^{-\frac{1}{2}}$$

$$\frac{a(1-\cos\theta)\, a\sin\theta\, d\theta}{\left(2a\,a(1-\cos\theta) - a^2(1-\cos\theta)^2\right)^{\frac{1}{2}}}$$

$$\frac{(a^2\sin\theta - a^2\cos\theta\sin\theta)\,d\theta}{\left(2a^2 - 2a^2\cos\theta - a^2(1-2\cos\theta+\cos^2\theta)\right)^{1/2}}$$

$$\frac{(a^2\sin\theta - a^2\cos\theta\sin\theta)\,d\theta}{\cancel{2}a^2 - \cancel{2a^2\cos\theta} - \cancel{a^2} + \cancel{2a^2\cos\theta} - a^2\cos^2\theta}$$

$$\frac{a^2\sin\theta\,(1-\cos\theta)\,d\theta}{(a^2 - a^2\cos^2\theta)^{\frac{1}{2}}}$$

$$= \frac{a^2\sin\theta\,(1-\cos\theta)\,d\theta}{\cancel{a}(1-\cos^2\theta)^{\frac{1}{2}}}$$

$$= \frac{a(1-\cos\theta)\,d\theta\,\cancel{\sin\theta}}{\cancel{(\sin^2\theta)^{\frac{1}{2}}}}$$

$$= a(1-\cos\theta).$$

6.4 The second form of the Euler Equation

-If the function f do not explicitly depends on x ;

-Second form of Euler equation may be derived as follow :

$$\frac{df}{dx} = \frac{d}{dx} f\{y, y'; x\}$$

$$= \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \frac{\partial f}{\partial x}$$

$$= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x}$$

$$\Longrightarrow \quad y''\frac{\partial f}{\partial y'} = \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} \qquad (1)$$

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \qquad (2)$$

-By using eq. (1) and (2) we get

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} - y'\frac{\partial f}{\partial y} + y'\frac{d}{dx}\frac{\partial f}{\partial y'}$$

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial x} + y'\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right)$$

-But from Euler first form $\left(\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y}\right) = 0$

$$\Rightarrow \frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0$$

-The second form can be written as follow

$$\Rightarrow f - y'\frac{\partial f}{\partial y'} = \text{constant} \qquad \left(\text{for } \frac{\partial f}{\partial x} = 0\right)$$

-Example 6.4 :

A *geodesic* is a line that represents the shortest path between any two points when the path is restricted to a particular surface. Find the geodesic on a sphere.

-Solution :

$$ds = \rho(d\theta^2 + \sin^2\theta\, d\phi^2)^{1/2} \tag{6.41}$$

The distance s between points 1 and 2 is therefore

$$s = \rho\int_1^2 \left[\left(\frac{d\theta}{d\phi}\right)^2 + \sin^2\theta\right]^{1/2} d\phi \tag{6.42}$$

and, if s is to be a minimum, f is identified as

$$f = (\theta'^2 + \sin^2\theta)^{1/2} \tag{6.43}$$

where $\theta' \equiv d\theta/d\phi$. Because $\partial f/\partial\phi = 0$, we may use the second form of the Euler equation (Equation 6.40), which yields

$$(\theta'^2 + \sin^2\theta)^{1/2} - \theta' \cdot \frac{\partial}{\partial\theta'}(\theta'^2 + \sin^2\theta)^{1/2} = \text{constant} \equiv a \tag{6.44}$$

Differentiating and multiplying through by f, we have

$$\sin^2\theta = a(\theta'^2 + \sin^2\theta)^{1/2} \tag{6.45}$$

This may be solved for $d\phi/d\theta = \theta'^{-1}$, with the result

$$\frac{d\phi}{d\theta} = \frac{a\csc^2\theta}{(1 - a^2\csc^2\theta)^{1/2}} \tag{6.46}$$

Solving for ϕ, we obtain

$$\phi = \sin^{-1}\left(\frac{\cot\theta}{\beta}\right) + \alpha \tag{6.47}$$

where α is the constant of integration and $\beta^2 \equiv (1 - a^2)/a^2$. Rewriting Equation 6.47 produces

$$\cot\theta = \beta\sin(\phi - \alpha) \tag{6.48}$$

To interpret this result, we convert the equation to rectangular coordinates by multiplying through by $\rho\sin\theta$ to obtain, on expanding $\sin(\phi - \alpha)$,

$$(\beta\cos\alpha)\rho\sin\theta\,\sin\phi - (\beta\sin\alpha)\rho\sin\theta\,\cos\phi = \rho\cos\theta \tag{6.49}$$

Because α and β are constants, we may write them as

$$\beta\cos\alpha \equiv A, \qquad \beta\sin\alpha \equiv B \tag{6.50}$$

Then Equation 6.49 becomes

$$A(\rho\sin\theta\,\sin\phi) - B(\rho\sin\theta\,\cos\phi) = (\rho\cos\theta) \tag{6.51}$$

The quantities in the parentheses are just the expressions for y, x, and z, respectively, in spherical coordinates (see Figure F-3, Appendix F); therefore Equation 6.51 may be written as

$$Ay - Bx = z \tag{6.52}$$

which is the equation of a plane passing through the center of the sphere. Hence the geodesic on a sphere is the path that the plane forms at the intersection with the surface of the sphere—a *great circle*. Note that the great circle is the maximum as well as the minimum "straight-line" distance between two points on the surface of a sphere.