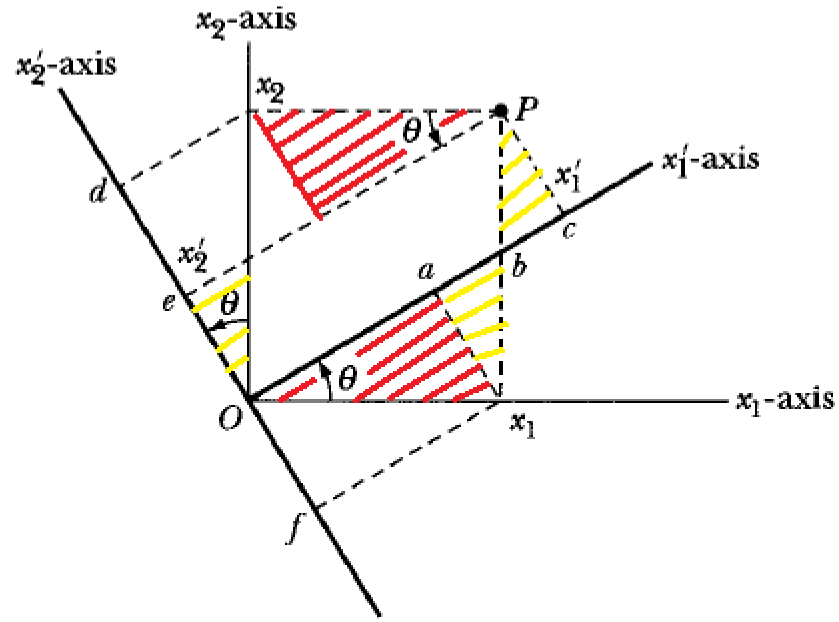
What is Classical Mechanics and why study it?

- Classical mechanics was the first branch of Physics to be discovered, and is the foundation upon which all other branches of Physics are built.
- *Classical mechanics* is the study of the *motion* of bodies
  - *Translational motion*
  - *Rotational motion*
  - *Oscillatory motion*
  - *Circular motion*
- Classical mechanics is known to fail in at least three ways.
  - At distances smaller than h/ mv ,
  - at velocities close to the speed of light
  - Only in Euclidean space and not in curved space

(1.3) Coordinate transformation

The new coordinate x_1' is the sum of the projection of x_1 onto the x_1'-axis (the line $\overline{Oa}$) plus (the line $\overline{ab} + \overline{bc}$)

$$x_1' = x_1 \cos\theta + x_2 \sin\theta$$
$$= x_1 \cos\theta + x_2 \cos\left(\frac{\pi}{2} - \theta\right) \quad \textbf{(1.2a)}$$



The coordinate x_2' is the sum of similar projections: $x_2' = \overline{Od} - \overline{de}$,

$$x_2' = -x_1 \sin\theta + x_2 \cos\theta$$
$$= x_1 \cos\left(\frac{\pi}{2} + \theta\right) + x_2 \cos\theta \quad \textbf{(1.2b)}$$

Let us introduce the following notation:

we define a set of numbers λ_{ij} by

$$\lambda_{ij} \equiv \cos(x'_i, x_j) \qquad (1.3)$$

Therefore, for Figure 1-2, we have

$$\left.\begin{aligned}
\lambda_{11} &= \cos(x'_1, x_1) = \cos\theta \\
\lambda_{12} &= \cos(x'_1, x_2) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \\
\lambda_{21} &= \cos(x'_2, x_1) = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta \\
\lambda_{22} &= \cos(x'_2, x_2) = \cos\theta
\end{aligned}\right\} \quad (1.4)$$

The equations of transformation (Equation 1.2) now become

$$x'_1 = x_1 \cos(x'_1, x_1) + x_2 \cos(x'_1, x_2)$$
$$= \lambda_{11} x_1 + \lambda_{12} x_2 \tag{1.5a}$$
$$x'_2 = x_1 \cos(x'_2, x_1) + x_2 \cos(x'_2, x_2)$$
$$= \lambda_{21} x_1 + \lambda_{22} x_2 \tag{1.5b}$$

Thus, in general, for three dimensions we have

$$\left.\begin{aligned} x'_1 &= \lambda_{11} x_1 + \lambda_{12} x_2 + \lambda_{13} x_3 \\ x'_2 &= \lambda_{21} x_1 + \lambda_{22} x_2 + \lambda_{23} x_3 \\ x'_3 &= \lambda_{31} x_1 + \lambda_{32} x_2 + \lambda_{33} x_3 \end{aligned}\right\} \tag{1.6}$$

or, in summation notation,

$$\boxed{x'_i = \sum_{j=1}^{3} \lambda_{ij} x_j, \quad i = 1, 2, 3} \tag{1.7}$$

The inverse transformation is

$$\boxed{x_i = \sum_{j=1}^{3} \lambda_{ji} x'_j, \quad i = 1, 2, 3} \tag{1.8}$$

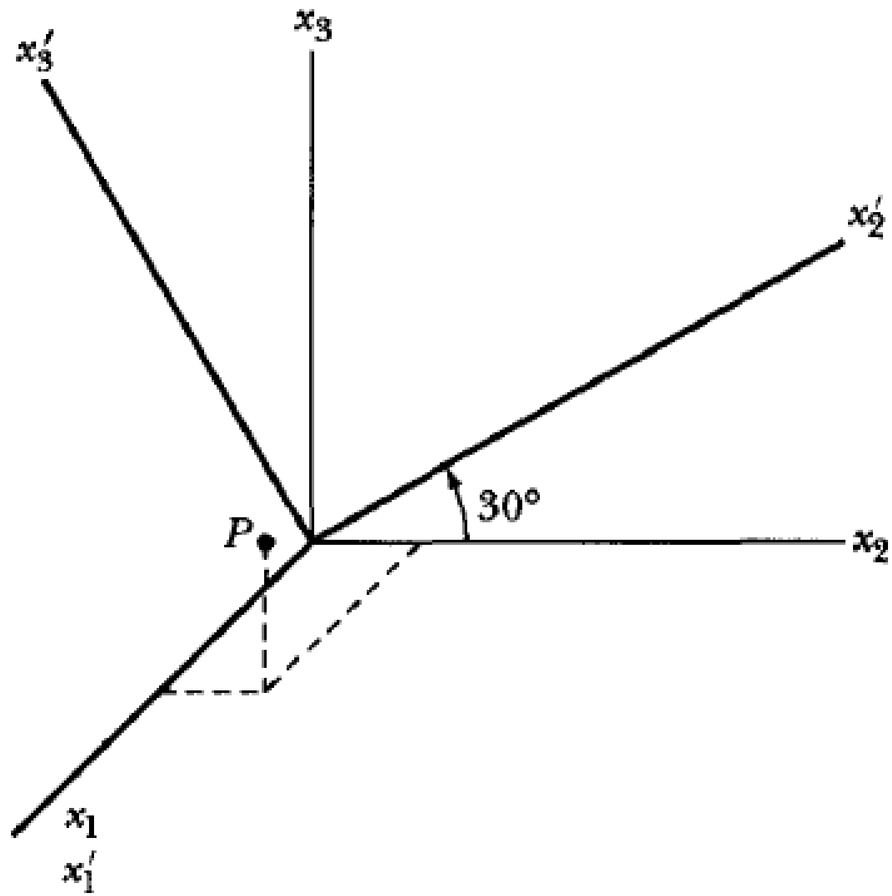
The quantity λ_{ij} is called the **direction cosine** of the x'_i-axis relative to the x_j-axis.

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \tag{1.9}$$

When $\boldsymbol{\lambda}$ is defined this way and when it specifies the transformation properties of the coordinates of a point, it is called a **transformation matrix** or a **rotation matrix**.

EXAMPLE 1.1

A point P is represented in the (x_1, x_2, x_3) system by $P(2, 1, 3)$. In another coordinate system, the same point is represented as $P(x_1', x_2', x_3')$ where x_2 has been rotated toward x_3 around the x_1-axis by an angle of 30° (Figure 1-3). Find the rotation matrix and determine $P(x_1', x_2', x_3')$.



Solution. The direction cosines λ_{ij} can be determined from Figure using the definition of Equation 1.3.

$$\lambda_{11} = \cos(x_1', x_1) = \cos(0°) = 1$$
$$\lambda_{12} = \cos(x_1', x_2) = \cos(90°) = 0$$
$$\lambda_{13} = \cos(x_1', x_3) = \cos(90°) = 0$$
$$\lambda_{21} = \cos(x_2', x_1) = \cos(90°) = 0$$
$$\lambda_{22} = \cos(x_2', x_2) = \cos(30°) = 0.866$$
$$\lambda_{23} = \cos(x_2', x_3) = \cos(90° - 30°) = \cos(60°) = 0.5$$
$$\lambda_{31} = \cos(x_3', x_1) = \cos(90°) = 0$$
$$\lambda_{32} = \cos(x_3', x_2) = \cos(90° + 30°) = -0.5$$
$$\lambda_{33} = \cos(x_3', x_3) = \cos(30°) = 0.866$$

$$\boldsymbol{\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.866 & 0.5 \\ 0 & -0.5 & 0.866 \end{pmatrix}$$

and using Equation 1.7, $P(x_1', x_2', x_3')$ is

$$
\begin{aligned}
x_1' &= \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3 = x_1 = 2 \\
x_2' &= \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 = 0.866x_2 + 0.5x_3 = 2.37 \\
x_3' &= \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 = -0.5x_2 + 0.866x_3 = 2.10
\end{aligned}
$$

Notice that the rotation operator preserves the length of the position vector.

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{x_1'^2 + x_2'^2 + x_3'^2} = 3.74$$

(1.4)Properties of rotation matrices

1- Transformation matrix λ specifying the rotation of any orthogonal coordinate system must then obey

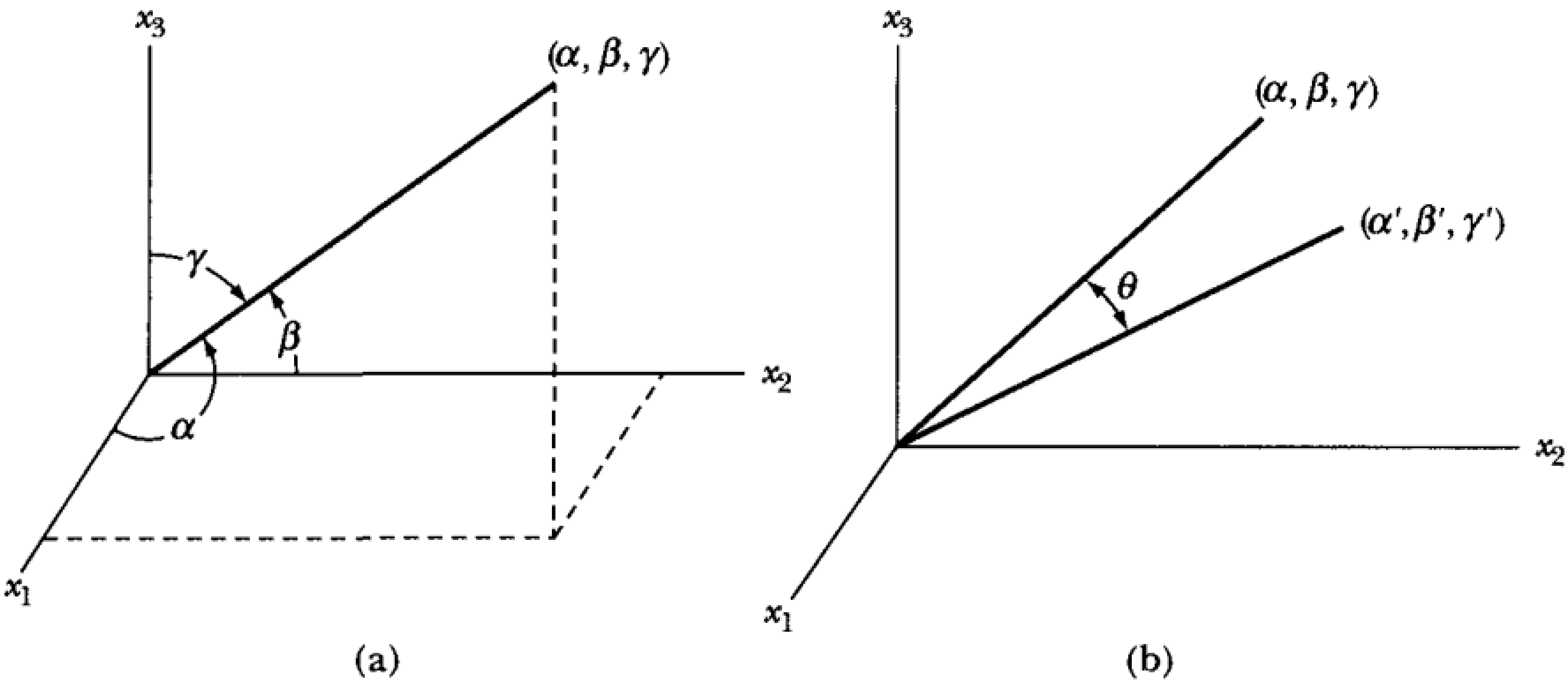
$$\sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}$$

- Let's prove that by recalling two trigonometric equations.

First : $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (1)$$

Second:

$$\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' \qquad (2)$$



(a) A line segment is defined by angles (α, β, γ) from the coordinate axes.
(b) Another line segment is added that is defined by angles $(\alpha', \beta', \gamma')$.

First, the x_1'-axis may be considered alone to be a line in the (x_1, x_2, x_3) coordinate system; the direction cosines of this line are $(\lambda_{11}, \lambda_{12}, \lambda_{13})$. Similarly, the direction cosines of the x_2'-axis in the (x_1, x_2, x_3) system are given by $(\lambda_{21}, \lambda_{22}, \lambda_{23})$.

Because the angle between the x_1'-axis and the x_2'-axis is $\pi/2$, we have, by using eq. 2

$$\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = \cos\theta = \cos(\pi/2) = 0$$

or

$$\sum_j \lambda_{1j}\,\lambda_{2j} = 0$$

And, in general,

$$\sum_j \lambda_{ij}\,\lambda_{kj} = 0, \quad i \neq k \tag{3}$$

Because the sum of the squares of the direction cosines of a line equals unity (Equation 1), we have for the x_1'-axis in the (x_1, x_2, x_3) system,

$$\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1$$

or

$$\sum_j \lambda_{1j}^2 = \sum_j \lambda_{1j}\lambda_{1j} = 1$$

and, in general,

$$\sum_j \lambda_{ij}\lambda_{kj} = 1, \quad i = k \qquad (4)$$

We may combine the results given by Equations 3 and 4 as

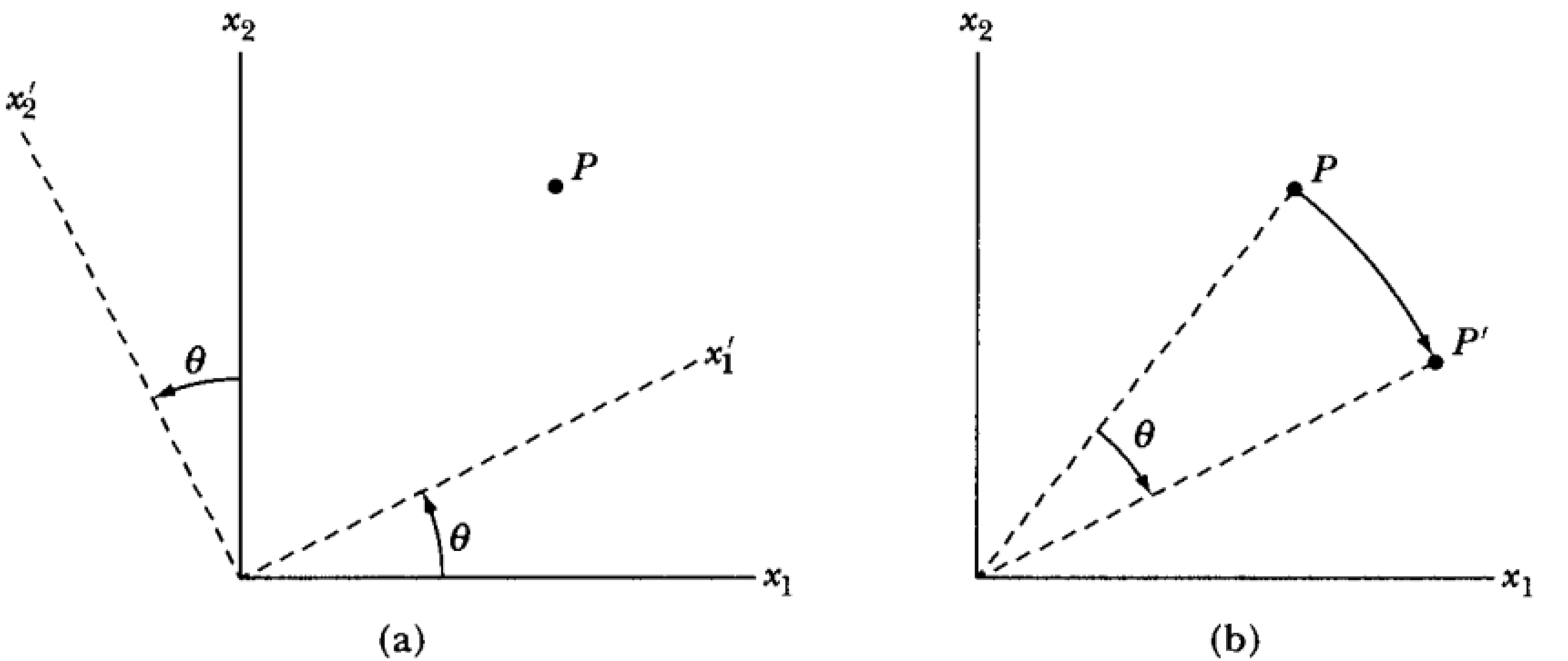
$$\boxed{\sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}} \qquad (5)$$

where δ_{ik} is the **Kronecker delta symbol**

$$\delta_{ik} = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases}$$

The validity of Equation (5) depends on the coordinate axes in each of the systems being mutually perpendicular.

Such systems are said to be **orthogonal**,
and Equation **5** is the **orthogonality condition**.



(a) The coordinate axes x_1, x_2 are rotated by angle θ, but the point P remains fixed. (b) In this case, the coordinates of point P are rotated to a new point P', but not the coordinate system.

2- we may elect to say either that the transformation acts on the *point* giving a new state of the point expressed with respect to a fixed coordinate system or that the transformation acts on the *frame of reference*

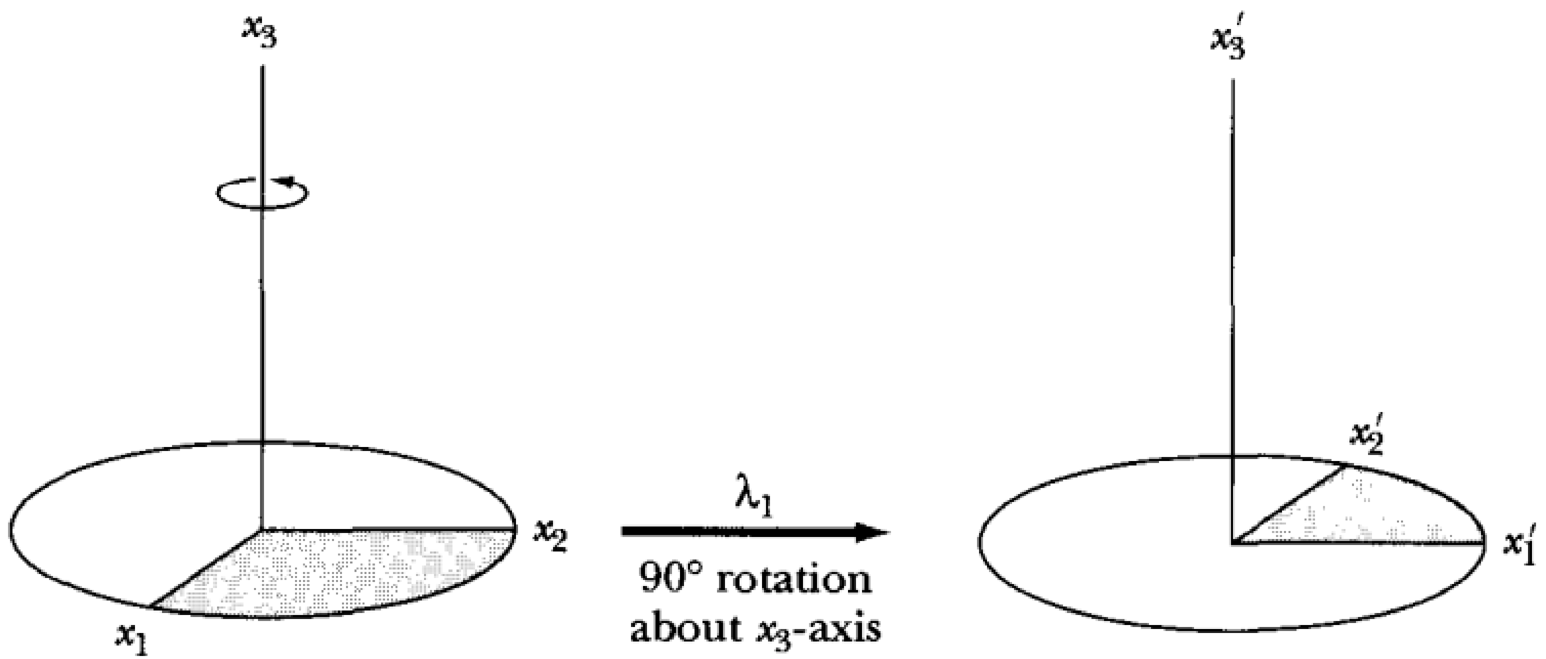
Mathematically, the interpretations are entirely equivalent.

1.7 – Geometrical significance of transformation matrices

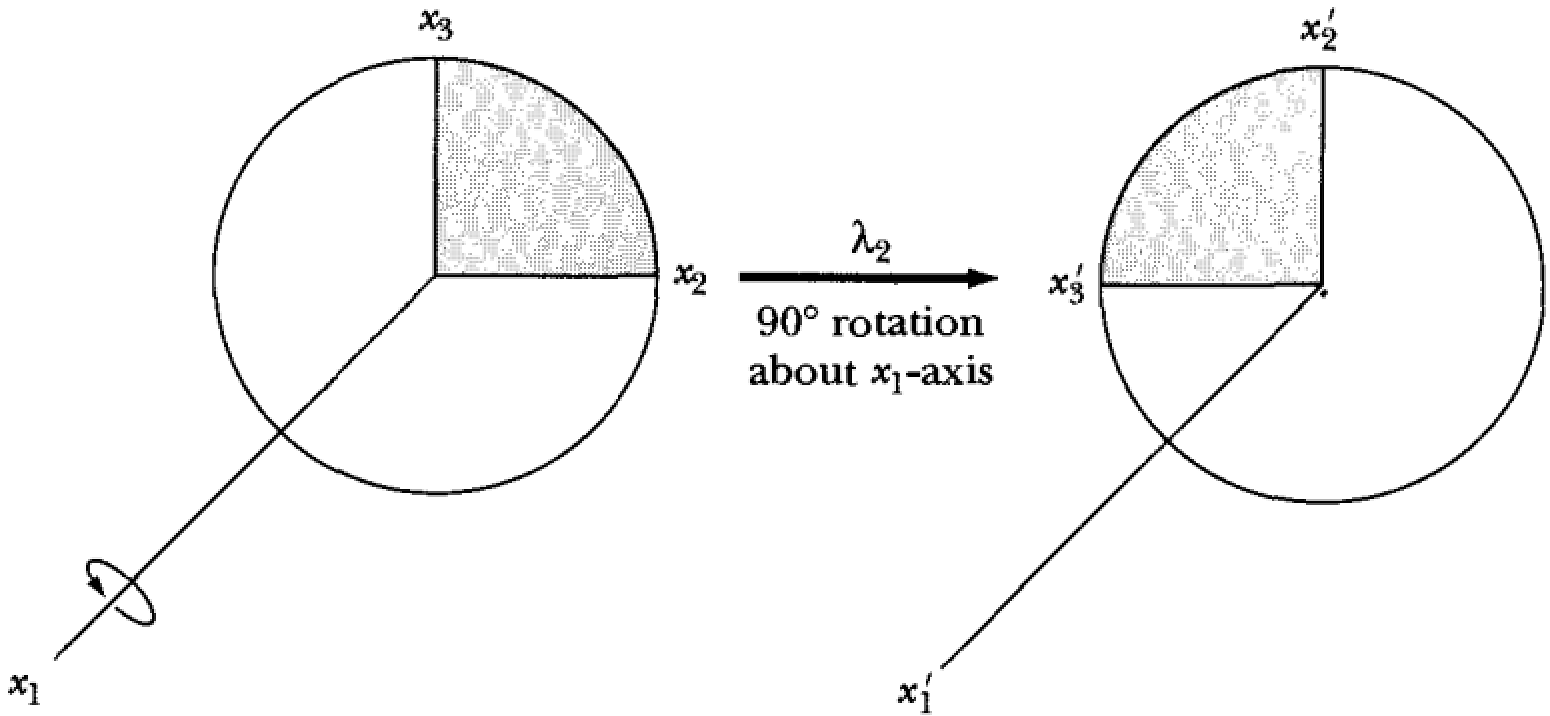
-For successive transformations, the order is important because the multiplication of rotation matrices is not commutative

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ rotation through } 90^\circ \text{ about the } x_3\text{-axis,}$$

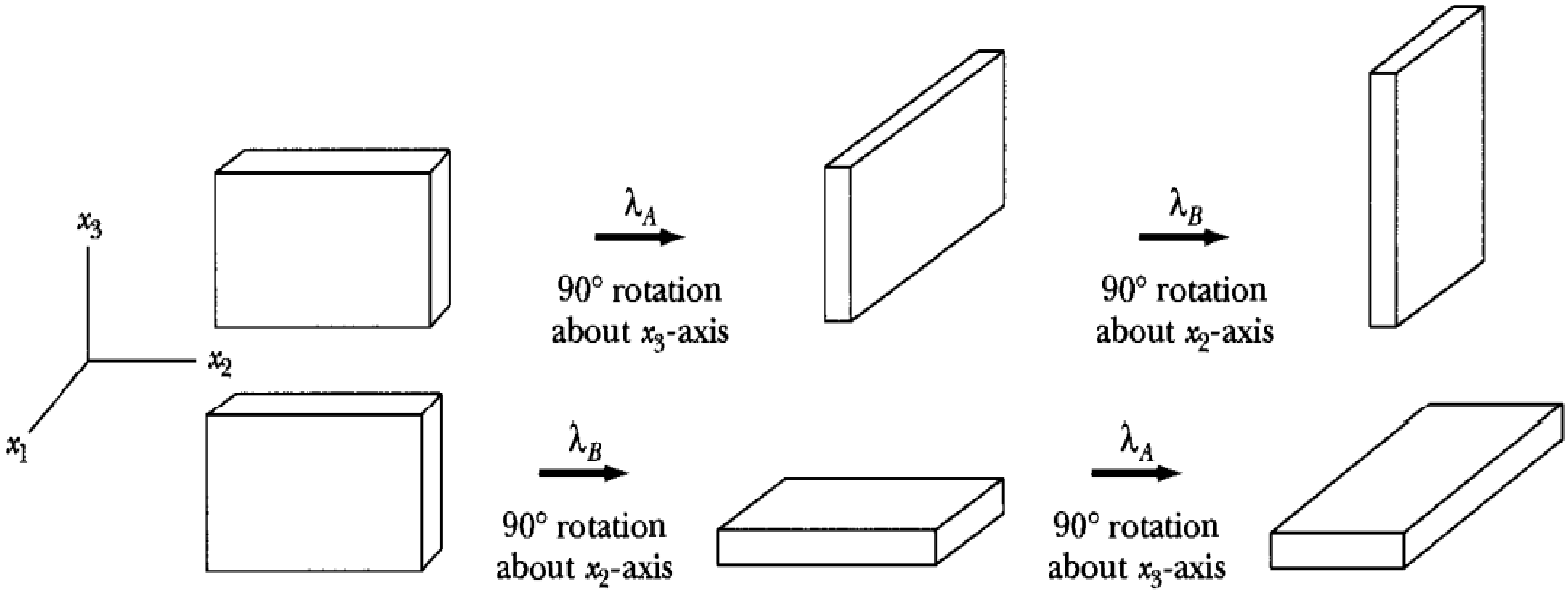
$$\boldsymbol{\lambda}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ rotation through } 90^\circ \text{ about the } x_1\text{-axis,}$$



Coordinate system x_1, x_2, x_3 is rotated 90° counter-clockwise (ccw) about the x_3-axis. This is consistent with the right-hand rule of rotation.



Coordinate system x_1, x_2, x_3 is rotated 90° ccw about the x_1-axis.



A parallelepiped undergoes two successive rotations in different order. The results are different.

$$\boldsymbol{\lambda}_3 = \boldsymbol{\lambda}_2\boldsymbol{\lambda}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{\lambda}_4 = \boldsymbol{\lambda}_1\boldsymbol{\lambda}_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \neq \boldsymbol{\lambda}_3$$

1.8 Defenations of a scalar and vector in terms of transformation properties

-If a quantity φ is unaffected under the following transformation

$$\text{with} \quad x_i' = \sum_j \lambda_{ij} x_j \qquad \sum_j \lambda_{ij} \lambda_{kj} = \delta_{ik}$$

-Then φ is called a Scalar.

-If a set of quantities (A1, A2, A3) is transformed as follows

$$A_i' = \sum_j \lambda_{ij} A_j$$

-Then the quantity A= (A1, A2, A3) is called a vector

1.14 Examples of derivatives – velocity and acceleration

-In rectangular coordinates, the expressions for ***r***, ***v***, and ***a*** are

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \sum_i x_i\mathbf{e}_i \quad \text{Position}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \sum_i \dot{x}_i\mathbf{e}_i = \sum_i \frac{dx_i}{dt}\mathbf{e}_i \quad \text{Velocity}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \sum_i \ddot{x}_i\mathbf{e}_i = \sum_i \frac{d^2x_i}{dt^2}\mathbf{e}_i \quad \text{Acceleration}$$

- v and a in polar coordinates take the following form :

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

-The expressions for ds, ds2, v2, and v are

Rectangular coordinates (x, y, z)

$$\left.\begin{aligned} d\mathbf{s} &= dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3 \\ ds^2 &= dx_1^2 + dx_2^2 + dx_3^2 \\ v^2 &= \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \\ \mathbf{v} &= \dot{x}_1\mathbf{e}_1 + \dot{x}_2\mathbf{e}_2 + \dot{x}_3\mathbf{e}_3 \end{aligned}\right\}$$

Spherical coordinates (r, θ, ϕ)

$$\left.\begin{aligned} d\mathbf{s} &= dr\mathbf{e}_r + r d\theta\mathbf{e}_\theta + r\sin\theta\, d\phi\mathbf{e}_\phi \\ ds^2 &= dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2 \\ v^2 &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 \\ \mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\sin\theta\,\dot{\phi}\mathbf{e}_\phi \end{aligned}\right\}$$

Cylindrical coordinates (r, ϕ, z)

$$\left.\begin{aligned} d\mathbf{s} &= dr\mathbf{e}_r + r d\phi\mathbf{e}_\phi + dz\mathbf{e}_z \\ ds^2 &= dr^2 + r^2 d\phi^2 + dz^2 \\ v^2 &= \dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2 \\ \mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z \end{aligned}\right\}$$

1.15 Angular velocity

-Angular velocity is defined as

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

-For motion in a circle of radius R, the magnitude of linear velocity is

$$v = R\frac{d\theta}{dt} = R\omega$$

-Or

$$v = r\omega \sin\alpha$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$