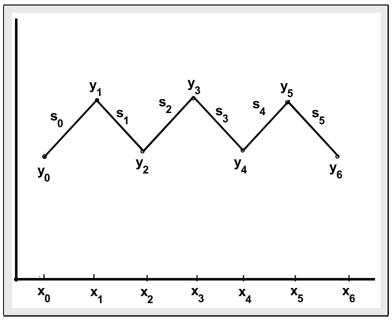

Figure 4.4: Linear spline.

4.3 Interpolation with Spline Functions

In the previous sections we studied the use of interpolation polynomials for approximating the values of the functions on closed intervals. An alternative approach is divide the interval into a collection of subintervals and construct a different approximating polynomial on each subinterval. Approximation by polynomial of this type is called *piecewise polynomial approximation*. Here, we will discuss some of the examples of a piecewise curve fitting techniques; the use of the *piecewise linear interpolation*.

Definition 4.1 (Spline Function)

Let $a = x_0 < x_1 < x_2 \cdots < x_n = b$. A function $s : [a, b] \to \mathbf{R}$ is a spline or spline function of degree m with points $x_0, x_1, \ldots, x_n$ if:
1. A function s is a piecewise polynomial such that, on each subinterval $[x_k, x_{k+1}]$, s has degree at most m.
2. A function s is $m-1$ times differentiable everywhere. •

A spline is a flexible drafting device that can be constrained to pass smoothly through a set of plotted data points. Spline functions are a mathematical tool which is an adaptation of this idea.

4.3.1 Piecewise Linear Interpolation

It is the one of the simplest piecewise polynomial interpolation for the approximation of the function, called *linear spline*. The linear spline is continuous function and the basic of it is simply connect consecutive points with straight lines. Consider the set of seven data points (x_0, y_0), (x_1, y_1), (x_2, y_2), $(x_3, y_3), (x_4, y_4), (x_5, y_5)$ and (x_6, y_6) which define six subintervals. These intervals are denoted as $[x_0, x_1], [x_1, x_2], [x_2, x_3]$, $[x_3, x_4], [x_4, x_5]$ and $[x_5, x_6]$, where $x_0, x_1, x_2, x_3, x_4, x_5$, and x_6 are distinct x-values. If we use a straight line on each subinterval (see Figure 4.4) then we can interpolate the data with a piecewise linear function, where

$$s_k(x) = p_k(x) = \frac{(x - x_{k+1})}{(x_k - x_{k+1})} y_k + \frac{(x - x_k)}{(x_{k+1} - x_k)} y_{k+1} = y_k + \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)} (x - x_k).$$

$$s_k(x) = A_k + B_k(x - x_k), \quad \text{where} \quad A_k = y_k \quad \text{and} \quad B_k = \frac{(y_{k+1} - y_k)}{(x_{k+1} - x_k)}. \tag{4.61}$$

Note that the linear spline must be continuous at given points $x_0, x_1, \ldots, x_n$ and

$$s(x_k) = f(x_k) = y_k, \qquad \text{for} \quad k = 0, 1, \ldots, n.$$

Example 4.48 *Find the values of unknown coefficients a and b so that the following function is a linear spline.*

$$s(x) = \begin{cases} a - x, & 0 \leq x \leq 1, \\ 3x - b, & 1 \leq x \leq 2, \\ 2x + 1, & 2 \leq x \leq 3. \end{cases}$$

Solution. *Since the given function is a linear spline, so s must be continuous at the internal points 1 and 2. Continuity at $x = 1$ implies that*

$$\begin{aligned} \lim_{x \to 1^-} s(x) &= \lim_{x \to 1^+} s(x), \\ \lim_{x \to 1^-} a - x &= \lim_{x \to 1^+} 3x - b, \\ a - 1 &= 3 - b, \end{aligned}$$

and it gives an equation of the from

$$a + b = 4.$$

Now continuity at $x = 2$ implies that

$$\begin{aligned} \lim_{x \to 2^-} s(x) &= \lim_{x \to 2^+} s(x), \\ \lim_{x \to 2^-} 3x - b &= \lim_{x \to 2^+} 2x + 1, \\ 6 - b &= 5, \end{aligned}$$

and it gives $b = 1$. Using this value of b, we get $a = 3$, and so

$$s(x) = \begin{cases} 3 - x, & 0 \leq x \leq 1, \\ 3x - 1, & 1 \leq x \leq 2, \\ 2x + 1, & 2 \leq x \leq 3, \end{cases}$$

is the linear spline function. •

Example 4.49 *Find the linear splines which interpolates the following data*

x_k	1	2	3	4
y_k	1.0	0.67	0.50	0.40

Find the approximation of the function $y(x) = \dfrac{2}{x+1}$ at $x = 2.9$. Compute absolute error.

Solution. *Given $x_0 = 1.0, x_1 = 2.0, x_2 = 3.0, x_3 = 4.0$, then using (??), we have*

$$A_0 = y_0 = 1.0, \; A_1 = y_1 = 0.67, \; A_2 = y_2 = 0.50, \; A_3 = y_3 = 0.4,$$

and

$$B_0 = \frac{(y_1 - y_0)}{(x_1 - x_0)} = \frac{(0.67 - 1.0)}{(2.0 - 1.0)} = -0.33,$$

$$B_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{(0.50 - 0.67)}{(3.0 - 2.0)} = -0.17,$$

$$B_2 = \frac{(y_3 - y_2)}{(x_3 - x_2)} = \frac{(0.40 - 0.50)}{(4.0 - 3.0)} = -0.10.$$

Now using (4.61), the linear splines for three subintervals are define as

$$s(x) = \begin{cases} s_0(x) = 1.0 - 0.33(x - 1.0) = 1.33 - 0.33x, & 1 \leq x \leq 2, \\ s_1(x) = 0.67 - 0.17(x - 2.0) = 1.01 - 0.17x, & 2 \leq x \leq 3, \\ s_2(x) = 0.50 - 0.10(x - 3.0) = 0.80 - 0.10x, & 3 \leq x \leq 4. \end{cases}$$

The $x = 2.9$ *lies in the interval* $[2, 3]$, *so,* $f(2.9) \approx s_1(2.9) = 1.01 - 0.17(2.9) = 0.517$, *and* $|f(2.9) - s_1(2.9)| = |0.513 - 0.517| = 0.004$, *is the required absolute error.* •

Using MATLAB command window, we can reproduce above results as follows:

$>> X = [1\ 2\ 3\ 4];\ Y = [1\ 0.67\ 0.50\ 0.40];\ x = 2.9; s = LSpline(X, Y, x)$

Program 4.4

MATLAB m-file for the Linear Spline Functions

```
function LS=LSpline(X,Y,x)
n=length(X); for i=n-1:-1:1
D = x - X(i); if (D >= 0); break; end; end
D = x - X(i); if (D < 0); i = 0; D = x - X(1); end
M = (Y(i + 1) - Y(i))/(X(i + 1) - X(i)); LS = Y(i) + M * D; end
```

Example 4.50 *For the following table find the value of* y *at* $x = 2.5$, *using piecewise linear interpolation*

x	1	2	3	4
y	35	40	65	72

Solution. *The point* $x = 2.5$ *lies between* 2 *and* 3. *Then*

$y(2.5) = \dfrac{(25 - 3)}{(2 - 3)}40 + \dfrac{(2.5 - 2)}{(3 - 2)}65 = 52.5.$ •

4.4 Exercises

1. Use the Lagrange interpolation formula based on the points $x_0 = 0, x_1 = 1, x_2 = 2.5$ to find the equation of the quadratic polynomial to approximate $f(x) = \dfrac{2}{x + 2}$ at $x = 2.3$.