MATH 244 Linear Algebra — Exam 1, Semester 481
Solution Key (Instructor Copy — Forms A, B, C, D)

Question 1 (10 pts, 1 pt each): Correct choices by form

Form	1	2	3	4	5	6	7	8	9	10
A	a	d	c	d	a	b	b	a	b	c
B	b	c	d	c	b	a	d	c	d	a
C	c	b	a	b	d	c	a	b	a	d
D	d	a	b	a	c	d	c	d	c	b

The correct statements are the same on every form; only their letters differ: (1) A^2 is upper triangular; (2) $A^2 + I$ is invertible; (3) (1, 2); (4) $A^{-1} = A + I$; (5) $\{(0, t, 0) : t \in \mathbb{R}\}$; (6) $x = 1$; (7) E adds −1 times row 1 to row 3; (8) AA^T; (9) $Ax = 0$ has infinitely many solutions; (10) 32.

Questions 2–5 are identical on all four forms.

Question 2 (5 pts)

Row reduce the augmented matrix [A | b]:

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -3 & m+5 & -1 \\ 1 & m-2 & 3 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & m-1 & -3 \\ 0 & m & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & m-1 & -3 \\ 0 & 0 & -m(m-1) & 3(m-1) \end{bmatrix}$$

($R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$, then $R_3 \to R_3 - mR_2$; the last column is the right-hand side.)

(i) Unique solution: $-m(m-1) \neq 0$, i.e. $m \neq 0$ and $m \neq 1$.

(ii) No solution: $m = 0$, since the last row becomes $0 = -3$.

(iii) Infinitely many solutions: $m = 1$, since the last row becomes $0 = 0$ and x_3 is free.

Question 3 (3 pts)

$R_2 \to R_2 + R_1$ and $R_3 \to R_3 - R_1$ make rows 2, 3 and 4 all equal to (0, −1, 3, −2). Eliminating rows 3 and 4 gives the echelon form

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so $x_3 = s$ and $x_4 = t$ are free, $x_2 = 3s - 2t$, and $x_1 = x_2 - 2x_3 - x_4 = s - 3t$. The general solution is

$$(x_1, x_2, x_3, x_4) = s\begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

Question 4 (3 pts)

(i) $(A^T)^{-1} = (A^{-1})^T =$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

(ii) $\det(A^{-1}) = 1\cdot(2 + 1) - 0 + 1\cdot(2 - 3) = 2$, so $\det A = 1/2$. Since $A^{-1} = \frac{1}{|A|}\operatorname{adj}(A)$, $\operatorname{adj} A = |A|\cdot A^{-1} = \tfrac{1}{2}A^{-1} =$

$$\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 1/2 & -1/2 \\ 3/2 & 1/2 & 1 \end{bmatrix}$$

Question 5 (4 pts)

(i) Cramer's rule applies exactly when the coefficient matrix has determinant nonzero:

$$\begin{vmatrix} \alpha & -1 & -1 \\ -1 & \alpha & 1 \\ 1 & 2 & 1 \end{vmatrix} = \alpha(\alpha - 2) - 2 + (\alpha + 2) = \alpha^2 - \alpha = \alpha(\alpha - 1).$$

So the system can be solved by Cramer's rule if and only if $\alpha \neq 0$ and $\alpha \neq 1$.

(ii) Replacing column 2 by the right-hand side $(-1, 1, 1)$ gives

$$y = \begin{vmatrix} \alpha & -1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} / (\alpha(\alpha - 1)) = 0 / (\alpha(\alpha - 1)) = 0,$$

because columns 2 and 3 of that determinant are equal.

Grading Rubric — Questions 2–5

***General policy**: deduct 0.5 for each arithmetic error, once only. Later steps that are correct for the student's own (wrong) values keep full marks. This is follow-through credit: the same error is never penalized twice. The 0.5 is taken from the criterion where the error occurs, not in addition to it, and no part scores below zero. Any correct method that is fully justified earns full credit. A correct final answer with no supporting work earns at most half the marks for that part.*

Question 2 (5 pts)

Criterion	Points
Reducing [A | b] to echelon form, with last row $(0, 0, -m(m-1) \mid 3(m-1))$	2
(i) Unique solution: $m \neq 0$ and $m \neq 1$, with a reason (e.g. $-m(m-1) \neq 0$ gives a pivot in every column)	1
(ii) No solution: $m = 0$, with a reason (the last row reads $0 = -3$)	1
(iii) Infinitely many solutions: $m = 1$, with a reason (the last row reads $0 = 0$, so x_3 is free)	1
Total	**5**

Partial credit:

Determinant method. Using $\det A = -m(m-1)$ to find the unique-solution case earns full credit for (i). For (ii) and (iii), the student must still check consistency separately at $m = 0$ and $m = 1$.

Cases without a reason. A correct value of m with no supporting reason (e.g. not showing the $0 = -3$ row) earns 0.5 of the 1 point for that case.

Swapped cases. Assigning $m = 0$ and $m = 1$ to the wrong cases earns 0 for (ii) and (iii), unless the swap follows correctly from an earlier arithmetic error (follow-through credit).

Question 3 (3 pts)

Criterion	Points
Correct row operations reducing the matrix to echelon form (rank 2: rows 2-4 become multiples of $(0, -1, 3, -2)$, leaving two zero rows)	1.5
Identifies two free variables (e.g. $x_3 = s$, $x_4 = t$) and back-substitutes correctly: $x_2 = 3s - 2t$, $x_1 = s - 3t$	1
States the general solution, e.g. $s(1, 3, 1, 0) + t(-3, -2, 0, 1)$, with $s, t \in \mathbb{R}$	0.5
Total	**3**

Partial credit:

Different free variables. Choosing other free variables (e.g. x_1, x_2) and getting an equivalent solution set earns full credit.

Only the trivial solution. Concluding that only $x = 0$ is a solution, or not introducing parameters, earns at most 1 point (for the elimination work).

Only one parameter. Losing a row during elimination and ending with a one-parameter family earns at most 1.5 points.

Question 4 (3 pts)

Criterion	Points
(i) Uses $(A^T)^{-1} = (A^{-1})^T$	0.5
(i) Writes the correct matrix: rows $(1, 2, 3)$, $(0, 1, 1)$, $(1, -1, 2)$	0.5
(ii) Computes $\det(A^{-1}) = 2$	0.5

Criterion	Points
(ii) Concludes $\det A = 1/\det(A^{-1}) = 1/2$	0.5
(ii) Uses $\text{adj } A = (\det A)\cdot A^{-1}$, or an equivalent valid method	0.5
(ii) Writes the correct matrix: $\frac{1}{2}A^{-1}$, with rows (1/2, 0, 1/2), (1, 1/2, −1/2), (3/2, 1/2, 1)	0.5
Total	**3**

Partial credit:

Computing A first. Finding A by inverting A^{-1} and then working from A earns full credit if correct. Deduct 0.5 per arithmetic error and give follow-through credit.

Determinant confusion. Writing $\text{adj } A = \det(A^{-1})\cdot A^{-1} = 2A^{-1}$ earns 1 of the 2 points for (ii), provided $\det(A^{-1}) = 2$ is correct and the formula structure is right.

Wrong matrix. Giving A^{-1} itself (forgetting the transpose), A, or A^T for (i) earns 0 for (i).

Question 5 (4 pts) — Cramer's rule

Criterion	Points
(i) States that Cramer's rule requires $\det A \neq 0$	0.5
(i) Computes $\det A = \alpha^2 - \alpha = \alpha(\alpha - 1)$ correctly	1
(i) Concludes $\alpha \neq 0$ and $\alpha \neq 1$	0.5
(ii) Sets up $y = \det(A_y)/\det A$, with column 2 replaced by (−1, 1, 1)	1
(ii) Evaluates $\det(A_y) = 0$, e.g. by noting that two columns are equal or by expanding	0.5
(ii) Concludes $y = 0$	0.5
Total	**4**

Partial credit:

Reversed condition. Answering $\alpha = 0$ or $\alpha = 1$ instead of $\alpha \neq 0, 1$, with a correct determinant, earns 1.5 of the 2 points for (i) if the student also states that $\det A \neq 0$ is required; otherwise it earns 1.

Other methods in (ii). Finding $y = 0$ by elimination, or by noticing that (0, 0, 1) is a solution, earns full credit for (ii) if justified (e.g. by checking that (0, 0, 1) satisfies all three equations). The question does not explicitly require Cramer's rule for y.

Wrong column replaced. Replacing column 1 or column 3 instead of column 2 earns 0.5 of the 2 points for (ii). The 0.5 for concluding $y = 0$ is not awarded, even though replacing column 1 also gives a zero determinant.

Follow-through from (i). A wrong $\det A$ carried into (ii) is not penalized again. Note that $y = 0$ holds whenever $\det A \neq 0$, because $\det(A_y) = 0$ for every α.