Integral Calculus

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(1) Indeterminate Forms

Definition

Let f be a defined function on an open interval I and $c \in I$ where f may not be defined at c. Then,

$$\lim_{x \to c} f(x) = L, \ L \in \mathbb{R}$$

means for every $\epsilon > 0$, there is $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Theorem

If $\lim_{x \longrightarrow c} f(x)$ and $\lim_{x \longrightarrow c} g(x)$ both exist, then

Sum Rule: $\lim_{x \longrightarrow c} (f(x) + g(x)) = \lim_{x \longrightarrow c} f(x) + \lim_{x \longrightarrow c} g(x)$.

Difference Rule: $\lim_{x \longrightarrow c} (f(x) - g(x)) = \lim_{x \longrightarrow c} f(x) - \lim_{x \longrightarrow c} g(x)$.

Product Rule: $\lim_{x \longrightarrow c} (f(x).g(x)) = \lim_{x \longrightarrow c} f(x) \times \lim_{x \longrightarrow c} g(x)$.

Constant Multiple Rule: $\lim_{x \longrightarrow c} (k \ f(x)) = k \ \lim_{x \longrightarrow c} f(x)$.

Quotient Rule: $\lim_{x \longrightarrow c} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \longrightarrow c} f(x)}{\lim_{x \longrightarrow c} g(x)}$.

Power Rule: $\lim_{x \longrightarrow c} (f(x))^{m/n} = \left(\lim_{x \longrightarrow c} f(x)\right)^{m/n}$.

Example

Find each limit if it exists.

1. $\lim_{x\to 1} x$
2. $\lim_{x\to 8} \sqrt{x}$
3. $\lim_{x\to 0}(x^2 - 2x + 1)$
4. $\lim_{x\to \pi} \sin\ x\ \cos\ x$
5. $\lim_{x\to 3^+} \frac{1}{(x-3)}$
6. $\lim_{x\to 1} \frac{x}{(x^2+1)}$

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6. $\lim_{x\to 1} \frac{x}{(x^2+1)}$

Solution:

1. $\lim_{x\to 1} x = 1$
2. $\lim_{x\to 8} \sqrt{x} = 2\sqrt{2}$
3. $\lim_{x\to 0} (x^2 - 2x + 1) = \lim_{x\to 0} x^2 - 2 \lim_{x\to 0} x + \lim_{x\to 0} 1 = 1.$
4. $\lim_{x\to \pi} \sin\ x\ \cos\ x = \lim_{x\to \pi} \sin\ x\ \lim_{x\to \pi} \cos\ x = 0$
5. $\lim_{x\to 3^+} \frac{1}{(x-3)} = \frac{\lim_{x\to 3^+} 1}{\lim_{x\to 3^+} (x-3)} = \infty$
6. $\lim_{x\to 1} \frac{x}{(x^2+1)} = \frac{\lim_{x\to 1} x}{\lim_{x\to 1} (x^2+1)} = \frac{1}{2}$

Example

Find each limit if it exists.

1. $\lim_{x\to 0} \frac{\sin\ x}{x} = \frac{0}{0}$
2. $\lim_{x\to\infty} \frac{e^x}{x} = \frac{\infty}{\infty}$
3. $\lim_{x\to 0^+} x^2 \ln x = 0.\infty$
4. $\lim_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \infty - \infty$

Example

Find each limit if it exists.

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4. $\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \infty - \infty$

In the following table, we categorize the indeterminate forms:

Case	Indeterminate Form
Quotient	$\frac{0}{0}$ and $\frac{\infty}{\infty}$
Product	$0.\infty$ and $0.(-\infty)$
Sum & Difference	$(-\infty) + \infty$ and $\infty - \infty$
Exponent	0^0, 1^∞, $1^{-\infty}$ and ∞^0

L'Hôpital's Rule

Theorem

Suppose f *and* g *are differentiable on an interval* I *and* $c \in I$ *where* f *and* g *may not be differentiable at* c. *If* $\frac{f(x)}{g(x)}$ *has the form* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$ *at* $x = c$ *and* $g'(x) \neq 0$ *for* $x \neq c$, *then*

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ *exists or equals to* ∞.

L'Hôpital's Rule

Theorem

Suppose f and g are differentiable on an interval I and $c \in I$ where f and g may not be differentiable at c. If $\frac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = c$ and $g'(x) \neq 0$ for $x \neq c$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or equals to ∞.

Remark

1. *L'Hôpital's rule works if $c = \pm\infty$ or when $x \to c^+$ or $x \to c^-$.*
2. *When applying L'Hôpital's rule, we should calculate the derivatives of $f(x)$ and $g(x)$ separately.*
3. *Sometimes, we need to apply L'Hôpital's rule twice.*

Example

Use L'Hôpital's rule to find each limit if it exists.

1. $\lim\limits_{x \to 5} \frac{\sqrt{x-1}-2}{x^2-25}$
2. $\lim\limits_{x \to 0} \frac{\sin\ x}{x}$
3. $\lim\limits_{x \to \infty} \frac{\ln x}{\sqrt{x}}$
4. $\lim\limits_{x \to \infty} \frac{e^x}{x}$

Example

Use L'Hôpital's rule to find each limit if it exists.

1. $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25}$
2. $\lim_{x\to 0} \frac{\sin\ x}{x}$
3. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$
4. $\lim_{x\to\infty} \frac{e^x}{x}$

Solution:

1. Since $\lim_{x\to 5} \sqrt{x-1}-2=0$ and $\lim_{x\to 5} x^2-2=0$, we have the indeterminate form $\frac{0}{0}$. By applying L'Hôpital's rule, we have
$$\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x^2-25} = \lim_{x\to 5} \frac{1}{4x\sqrt{x-1}} = \frac{1}{40}.$$
2. The quotient has the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule to have
$$\lim_{x\to 0} \frac{\sin\ x}{x} = \lim_{x\to 0} \frac{\cos\ x}{1} = 1.$$
3. The indeterminate form is $\frac{\infty}{\infty}$. Apply L'Hôpital's rule to obtain
$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$
4. The indeterminate form is $\frac{\infty}{\infty}$. By applying L'Hôpital's rule, we have
$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty.$$

Techniques for finding the limits of other indeterminate forms:

- Indeterminate form $0.\infty$.
  1. Write $f(x)\,g(x)$ as $\frac{f(x)}{1/g(x)}$ or $\frac{g(x)}{1/f(x)}$.
  2. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

- Indeterminate form $\infty - \infty$.
  1. Write the form as a quotient or product.
  2. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

- Indeterminate forms 0^0, 1^∞, $1^{-\infty}$ or ∞^0.
  1. Let $y = f(x)^{g(x)}$
  2. Take the natural logarithm $\ln y = \ln f(x)^{g(x)} = g(x) \ln f(x)$.
  3. Apply L'Hôpital's rule to the resulting indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example

Find each limit if it exists.

1. $\lim\limits_{x\to 0^+} x^2 \ln x$
2. $\lim\limits_{x\to \frac{\pi}{4}} (1 - \tan\ x)\ \sec\ 2x$
3. $\lim\limits_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$
4. $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}}$

Solution:

1) The indeterminate form is $0.(-\infty)$, so we cannot apply L'Hôpital's rule. We need to rearrange the expression in a way that enables us to apply L'Hôpital's rule. By using the previous techniques, we obtain

$$x^2 \ln x = \frac{\ln x}{\frac{1}{x^2}}.$$

The limit of the new expression is of the form $\frac{\infty}{\infty}$. Therefore, we can apply L'Hôpital's rule:

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

Hence, $\lim_{x\to 0^+} x^2 \ln x = 0$.

Solution:

1) The indeterminate form is $0.(-\infty)$, so we cannot apply L'Hôpital's rule. We need to rearrange the expression in a way that enables us to apply L'Hôpital's rule. By using the previous techniques, we obtain

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$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{-2} = 0.$$

Hence, $\lim\limits_{x\to 0^+} x^2 \ln x = 0$.

2) The indeterminate form is $0.\infty$, so we try to rewrite the function to apply L'Hôpital's rule. We know that $\sec\ x = 1/\cos\ x$, thus

$$(1-\tan\ x)\sec\ 2x = \frac{(1-\tan\ x)}{\cos\ 2x}.$$

Now, the limit of the new expression is of the form $\frac{0}{0}$. From L'Hôpital's rule, we have

$$\lim_{x\to \frac{\pi}{4}} \frac{(1-\tan\ x)}{\cos\ 2x} = \lim_{x\to \frac{\pi}{4}} \frac{\sec^2\ x}{2\sin\ 2x} \qquad \text{(L'Hôpital's rule)}$$

$$= \frac{(\sqrt{2})^2}{2} = 1.$$

Hence, $\lim\limits_{x\to \frac{\pi}{4}} (1-\tan\ x)\ \sec\ 2x = 1$.

3) The indeterminate form is $\infty - \infty$. To treat this form, we write the function as a single fraction

$$\frac{1}{x-1} - \frac{1}{\ln x} = \frac{\ln x - x + 1}{(x-1)\ln x}.$$

The new expression takes the indeterminate form $\frac{0}{0}$. From L'Hôpital's rule,

$$\lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{1-x}{x\ \ln x + x - 1}.$$

We have the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule again to have

$$\lim_{x\to 1^+} \frac{1-x}{x\ \ln x + x - 1} = \lim_{x\to 1^+} \frac{-1}{\ln x + 2} = \frac{-1}{2}.$$

Hence, $\lim\limits_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = -\frac{1}{2}$.

3) The indeterminate form is $\infty - \infty$. To treat this form, we write the function as a single fraction

$$\frac{1}{x-1} - \frac{1}{\ln x} = \frac{\ln x - x + 1}{(x-1)\ln x}.$$

The new expression takes the indeterminate form $\frac{0}{0}$. From L'Hôpital's rule,

$$\lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} = \lim_{x\to 1^+} \frac{1-x}{x\ \ln x + x - 1}.$$

We have the indeterminate form $\frac{0}{0}$. We apply L'Hôpital's rule again to have

$$\lim_{x\to 1^+} \frac{1-x}{x\ \ln x + x - 1} = \lim_{x\to 1^+} \frac{-1}{\ln x + 2} = \frac{-1}{2}.$$

Hence, $\lim\limits_{x\to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = -\frac{1}{2}$.

4) The limit is of the form 1^∞. To treat this form, let $y = (1+x)^{\frac{1}{x}}$. By taking the natural logarithm of both sides, we have

$$\ln y = \frac{1}{x}\ln(1+x)$$

$$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0} \frac{\ln(1+x)}{x}.$$

The indeterminate form is $\frac{0}{0}$. By applying L'Hôpital's rule, we obtain

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

Hence,

$$\lim_{x\to 0} \ln y = 1 \Rightarrow e^{\lim_{x\to 0} \ln y} = e^1 \qquad \text{(take the natural exponential function of both sides)}$$

$$\Rightarrow \lim_{x\to 0} e^{(\ln y)} = e$$

$$\Rightarrow \lim_{x\to 0} y = e$$

$$\Rightarrow \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

Hence,

$$\lim_{x\to 0} \ln y = 1 \Rightarrow e^{\lim\limits_{x\to 0} \ln y} = e^1$$ (take the natural exponential function of both sides)

$$\Rightarrow \lim_{x\to 0} e^{(\ln y)} = e$$

$$\Rightarrow \lim_{x\to 0} y = e$$

$$\Rightarrow \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$$

(1) Improper Integrals

Definition

The integral $\int_a^b f(x)\, dx$ is called a proper integral if

1. *the interval $[a, b]$ is finite and closed, and*
2. *$f(x)$ is defined on $[a, b]$.*

If condition 1 or 2 is not satisfied, the integral is improper. In the following, we discuss the improper integrals.

(A) Infinite Intervals

Definition

1. *Let f be a continuous function on $[a, \infty)$. The improper integral* $\int_a^{\infty} f(x)\ dx$ *is defined as follows:*

$$\int_a^{\infty} f(x)\ dx = \lim_{t \to \infty} \int_a^t f(x)\ dx \quad \text{if the limit exists.}$$

2. *Let f be a continuous function on $(-\infty, b]$. The improper integral* $\int_{-\infty}^{b} f(x)\ dx$ *is defined as follows:*

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{t \to -\infty} \int_t^b f(x)\ dx \quad \text{if the limit exists.}$$

The previous integrals are convergent (or to converge) if the limit exists as a finite number. However, if the limit does not exist or equals $\pm\infty$, the integral is called divergent (or to diverge).

3. *Let f be a continuous function on $\mathbb{R}$ and $a \in \mathbb{R}$. The improper integral* $\int_{-\infty}^{\infty} f(x)\ dx$ *is defined as follows:*

$$\int_{-\infty}^{\infty} f(x)\ dx = \int_{-\infty}^{a} f(x)\ dx + \int_a^{\infty} f(x)\ dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the

Note:

1. If an improper integral is convergent, the value of the integral is the value of the limit.
2. If both integrals in item 3 converge, then the value of the improper integral is the sum of values of the two integrals.

Example

Determine whether the integral converges or diverges.

1. $\displaystyle\int_0^{\infty} \frac{1}{(x+2)^2}\, dx$
2. $\displaystyle\int_0^{\infty} \frac{x}{1+x^2}\, dx$
3. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

Note:

1. If an improper integral is convergent, the value of the integral is the value of the limit.
2. If both integrals in item 3 converge, then the value of the improper integral is the sum of values of the two integrals.

Example

Determine whether the integral converges or diverges.

1. $\displaystyle\int_0^\infty \frac{1}{(x+2)^2}\,dx$
2. $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,dx$
3. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution:

1) $\displaystyle\int_0^\infty \frac{1}{(x+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx.$

The integral

$$\int_0^t \frac{1}{(x+2)^2}\,dx = \int_0^t (x+2)^{-2}\,dx = \left[\frac{-1}{x+2}\right]_0^t = -\left(\frac{1}{t+2}-\frac{1}{2}\right).$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{1}{(x+2)^2}\,dx = -\lim_{t\to\infty}\left(\frac{1}{t+2}-\frac{1}{2}\right) = -\left(0-\frac{1}{2}\right) = \frac{1}{2}.$$

This implies that the integral converges and has the value $\frac{1}{2}$.

2) $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx.$
The integral

$$\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\Big[\ln(1+x^2)\Big]_0^t = \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(1+t^2).$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\lim_{t\to\infty}\ln(1+t^2) = \infty.$$

The improper integral diverges.

2) $\int_0^\infty \frac{x}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx.$
The integral

$$\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\Big[\ln(1+x^2)\Big]_0^t = \frac{1}{2}\ln(1+t^2) - \frac{1}{2}\ln(1) = \frac{1}{2}\ln(1+t^2).$$

Thus,

$$\lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx = \frac{1}{2}\lim_{t\to\infty}\ln(1+t^2) = \infty.$$

The improper integral diverges.

3) $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\,dx.$

We know that $\int \frac{1}{1+x^2}\,dx = \tan^{-1}\ x + c$, so

$$\begin{aligned}\lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2} &= \lim_{t\to-\infty}\left[0-\tan^{-1}(t)\right] + \lim_{t\to\infty}\left[\tan^{-1}\ t - 0\right]\\ &= -\lim_{t\to-\infty}\tan^{-1}\ t + \lim_{t\to\infty}\tan^{-1}\ t\\ &= -(-\frac{\pi}{2}) + \frac{\pi}{2} = \pi.\end{aligned}$$

The integral is convergent and has the value π.

(B) Discontinuous Integrands

Definition

1. *If* f *is continuous on* $[a, b)$ *and has an infinite discontinuity at* b *i.e.,* $\lim_{x \to b^-} f(x) = \pm\infty$, *then*

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx \quad \text{if the limit exists.}$$

2. *If* f *is continuous on* $(a, b]$ *and has an infinite discontinuity at* a *i.e.,* $\lim_{x \to a^+} f(x) = \pm\infty$, *then*

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^a f(x)\, dx \quad \text{if the limit exists.}$$

In items 1 and 2, the integral is convergent if the limit exists as a finite number; otherwise the integral is divergent.

3. *If* f *is continuous on* $[a, b]$ *except at* $c \in (a, b)$ *such that* $\lim_{x \to c^\pm} f(x) = \pm\infty$, *the improper integral* $\int_a^b f(x)\, dx$ *is defined as follows:*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

The integral is convergent if both integrals on the right side are convergent; otherwise the integral is divergent.

Example

Determine whether the integral converges or diverges.

1. $\displaystyle\int_0^4 \frac{1}{(4-x)^{\frac{3}{2}}}\,dx$
2. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x}{\sqrt{\sin x}}\,dx$
3. $\displaystyle\int_{-3}^{1} \frac{1}{x^2}\,dx$