Independence of Path and Conservative Vector Field

Mongi BLEL

King Saud University

March 25, 2024

1 Independence of Path

Definition

We say that the line integral $\int_C F.d\mathbf{r}$ is independent of path in the domain D if the integral is the same for every path contained in D that has the same beginning and ending points.

Theorem

Let $F = (f, g, h)$ be a continuous vector field defined on a connected region D and let C be a smooth parametric curve on D parameterized by $C(t) = (x(t), y(t), z(t))$, $t \in [a, b]$.
The integral

$$\begin{aligned} \int_C F.d\mathbf{r} &= \int_a^b f(x(t), y(t), z(t))x'(t)dt + \int_a^b f(x(t), y(t), z(t))x'(t)dt \\ &\int_a^b f(x(t), y(t), z(t))x'(t)dt \end{aligned}$$

is independent of the path if and only if F is conservative.

Theorem

(Fundamental Theorem of Line Integrals)
Consider a smooth parametric curve C parameterized by a smooth vector function $C(t) = (x(t), y(t), z(t))$, $t \in [a, b]$. If f is a continuously differentiable function on a domain containing the curve C, then $\int_C \nabla f . d\mathbf{r} = f(C(b)) - f(C(a))$.
In particular, if the curve is closed, (i.e. $C(b) = C(a)$), then $\int_C \nabla f . d\mathbf{r} = 0$.

Example

Consider the vector field $F(x, y) = (2xy - 3, x^2 + 4y^3 + 5)$.
The line integral $\int_C F.d\mathbf{r}$ is independent of path. Then, evaluate the line integral for any curve C with initial point at $(-1, 2)$ and terminal point at $(2, 3)$.
$F = \nabla f$, $\frac{\partial f}{\partial x} = 2xy - 3$, $f = x^2y - 3x + g(y)$,
$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 + 4y^3 + 5$. Then $f = x^2y - 3x + y^4 + 5y$.
$\int_C F.d\mathbf{r} = f(2, 3) - f(-1, 2) = 102 - 31 = 71$.

Conservative Vector Fields

Let $F(x, y) = (M(x, y), N(x, y))$, where we assume that $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on an open, simply-connected region $D \subset \mathbb{R}^2$. The following five statements are equivalent, meaning that for a given vector field, either all five statements are true or all five statements are false.

1. $F(x, y)$ is conservative on D.
2. $F(x, y)$ is a gradient field in D (i.e., $F(x, y) = \nabla f(x, y)$, for some potential function f, for all $(x, y) \in D$).
3. $\displaystyle\int_C F.d\mathbf{r}$ is independent of path in D.
4. $\displaystyle\int_C F.d\mathbf{r} = 0$ for every piecewise-smooth closed curve C lying in D.
5. $\dfrac{\partial M}{\partial y}(x, y) = \dfrac{\partial N}{\partial x}(x, y)$, for all $(x, y) \in D$.

Theorem

Consider a simple connected region D and let F be a vector field defined on D.
The following properties of a vector field F are equivalent:

1. F is conservative.
2. $\displaystyle\int_C F.d\mathbf{r}$ is path-independent, (i.e. meaning that it only depends on the endpoints of the curve C.
3. $\displaystyle\oint_C F.d\mathbf{r} = 0$ around any closed smooth curve C in D.