The Inclusion-Exclusion Principle

The sum principle for counting is the simplest of the basic counting principles. It states that if we have $A_1, \ldots, A_n$ finite sets that are pairwise disjoint, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$. The Inclusion-Exclusion Principle, in its simplest form, gives us a formula to calculate $|A_1 \cup A_2 \cup \cdots \cup A_n|$ when we allow the sets $A_1, \ldots, A_n$ to overlap.

In what follows, we will assume that U is a given finite universal set and that $A_1, \ldots, A_n$ are subsets of U; for each $i = 1, 2, \ldots, n$, we define α_i as follows:

$$\alpha_i = \sum_{\{j_1, j_2, \ldots, j_i\} \subseteq \{1,2,\ldots,n\}} |A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_i}| \tag{1}$$

where the sum is taken over all possible subsets of indices $\{j_1, j_2, \ldots, j_i\}$ from $\{1, 2, \ldots, n\}$.

0.1 Theorem (The Inclusion-Exclusion Principle)

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \alpha_1 - \alpha_2 + \cdots + (-1)^{n-1}\alpha_n \tag{2}$$

1 Proof

Let $x \in A_1 \cup A_2 \cup \cdots \cup A_n$. When calculating $|A_1 \cup A_2 \cup \cdots \cup A_n|$ the element x is counted once, and it is different when calculating $\alpha_1 - \alpha_2 + \cdots + (-1)^{n-1}\alpha_n$. We will prove that the contribution of x to the calculation of this number is equal to 1. We assume that x belongs only to the sets $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$. Thus, the contribution of x to the calculation of the number $\alpha_1 = |A_1| + |A_2| + \cdots + |A_n|$ is equal to m. Also, the contribution of x to the calculation of the number $\alpha_2 = |A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|$ is equal to $\binom{m}{2}$ because the contribution of x to the calculation of $|A_i \cap A_j|$ is 0 if $\{i, j\} \not\subseteq \{i_1, \ldots, i_m\}$ and is 1 if $\{i, j\} \subseteq \{i_1, \ldots, i_m\}$. Similarly, the contribution of x to the calculation of the number α_k is equal to $\binom{m}{k}$ for every $1 \le k \le n$. Thus, the contribution of x to the calculation of the number $\alpha_1 - \alpha_2 + \cdots + (-1)^{n-1}\alpha_n$ is equal to

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots + (-1)^{m-1}\binom{m}{m} = 1 - (1 - \binom{m}{0} + \binom{m}{1} - \cdots + (-1)^m \binom{m}{m})$$

We know from the Binomial Theorem that

$$\binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \cdots + (-1)^m \binom{m}{m} = (1-1)^m = 0$$

and therefore

$$\binom{m}{1} - \binom{m}{2} + \cdots + (-1)^{m-1}\binom{m}{m} = \binom{m}{0} = 1.$$

This completes the proof.

In many problems, we calculate the number of elements that do not belong to any of the sets $A_1, A_2, \ldots, A_n$ using the following result of the Inclusion-Exclusion Principle.

1.1 Corollary

If U is a finite universal set and $A_1, \ldots, A_n$ are subsets of U, then

$$|U - (A_1 \cup A_2 \cup \cdots \cup A_n)| = |U| - \alpha_1 + \alpha_2 - \cdots + (-1)^n \alpha_n$$

1.2 Example

Find the number of integers x such that $1 \leq x \leq 500$, where x is not divisible by 5, not divisible by 6, and not divisible by 8.

1.3 Solution

Let $U = \{1, 2, ..., 500\}$, and let $A_1 = \{x \in U : 5|x\}$, $A_2 = \{x \in U : 6|x\}$, and $A_3 = \{x \in U : 8|x\}$. We want to calculate the number $|U - (A_1 \cup A_2 \cup A_3)|$. We note that: $|A_1| = \lfloor \frac{500}{5} \rfloor = 100$, $|A_2| = \lfloor \frac{500}{6} \rfloor = 83$, $|A_3| = \lfloor \frac{500}{8} \rfloor = 62$. As is known, $a|n$ and $b|n$ if and only if $\operatorname{lcm}(a, b)|n$. Therefore, we find: $|A_1 \cap A_2| = \lfloor \frac{500}{\operatorname{lcm}(5,6)} \rfloor = \lfloor \frac{500}{30} \rfloor = 16$. $|A_1 \cap A_3| = \lfloor \frac{500}{\operatorname{lcm}(5,8)} \rfloor = \lfloor \frac{500}{40} \rfloor = 12$. $|A_2 \cap A_3| = \lfloor \frac{500}{\operatorname{lcm}(6,8)} \rfloor = \lfloor \frac{500}{24} \rfloor = 20$. And $|A_1 \cap A_2 \cap A_3| = \lfloor \frac{500}{\operatorname{lcm}(5,6,8)} \rfloor = \lfloor \frac{500}{120} \rfloor = 4$. Therefore: $|U - (A_1 \cup A_2 \cup A_3)| = |U| - (\alpha_1 - \alpha_2 + \alpha_3) = |U| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3| = 500 - (100 + 83 + 62) + (16 + 12 + 20) - 4 = 299$.

2 Example

Find the number of integer solutions for the equation $X_1 + X_2 + X_3 = 13$ with the conditions $0 \leq X_1 \leq 6$, $0 \leq X_2 \leq 9$, and $0 \leq X_3 \leq 3$.

3 Solution

Let U be the set of integer solutions with $X_i \geq 0$ for each $i = 1, 2, 3$. Let A_1 be the set of integer solutions with $X_1 \geq 7, X_2 \geq 0, X_3 \geq 0$. Let A_2 be the set of integer solutions with $X_1 \geq 0, X_2 \geq 10, X_3 \geq 0$. Let A_3 be the set of integer solutions with $X_1 \geq 0, X_2 \geq 0, X_3 \geq 4$. We need to calculate the number $|U - (A_1 \cup A_2 \cup A_3)|$.

It is clear that $|U| = \binom{3-1+13}{13} = \binom{15}{13} = 105$. Similarly, we find that $|A_1| = \binom{3-1+13-7}{13-7} = \binom{8}{6} = 28$. $|A_2| = \binom{3-1+13-10}{13-10} = \binom{5}{3} = 10$. $|A_3| = \binom{3-1+13-4}{13-4} = \binom{11}{9} = 55$.

$$|A_1 \cap A_2| = 0, \quad |A_3| = \binom{3-1+13-4}{13-4} = 55$$

$$|A_1 \cap A_2 \cap A_3| = 0, \quad |A_2 \cap A_3| = \binom{3-1+13-7-4}{13-7-4} = 6$$

$$|A_1 \cap A_3| = \binom{3-1+13-7}{13-7} = 28$$

Therefore,

$$|U - (A_1 \cup A_2 \cup A_3)| = 105 - (28 + 10 + 55) + (0 + 6 + 0) - 0 = 18$$

From the definition of the union of sets, it follows that the Inclusion-Exclusion Principle gives the number of elements that belong to at least one of the sets $A_1, A_2, \ldots, A_n$. To obtain two simple generalizations of this principle, we denote the number of elements that belong to **exactly** m of the sets $A_1, \ldots, A_n$ by e_m, and we use the symbol $N_{\geq m}$ to denote the number of elements that belong to **at least** m of the sets $A_1, \ldots, A_n$. The following theorem gives us the two required generalizations.

4 Theorem

1. $e_m = N_{\geq m} - \binom{m+1}{m} N_{\geq m+1} + \binom{m+2}{m} N_{\geq m+2} - \cdots + (-1)^{n-m} \binom{n}{m} N_{\geq n}$
2. $N_{\geq m} = \sum_{k=m}^{n} (-1)^{k-m} \binom{k-1}{m-1} e_k$

5 Proof

We assume that x belongs to exactly k sets.