CHAPTER 9

Inverse Trigonometric, Hyperbolic, and Inverse Hyperbolic Functions

[Calculus is] the outcome of a dramatic intellectual struggle which has lasted for twenty-five hundred years . . .

RICHARD COURANT

Mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true.

BERTRAND RUSSELL

Do I contradict myself?
Very well then I contradict myself.
(I am large, I contain multitudes.)

WALT WHITMAN (1819–1892)

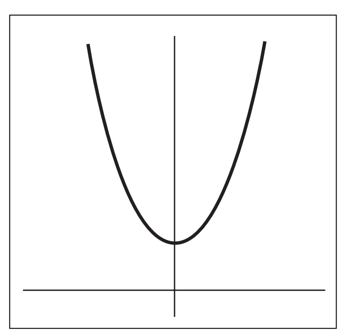
CAREERS IN CALCULUS: ARCHITECTURE

In thinking about the skills needed by a good architect, most people would include drawing ability, an eye for art and design, and a good grasp of civil engineering. But did you know that mathematics forms the backbone of architectural theory and practice? In order to design a house, for example, or a store, an architect needs to be able to plan, calculate, and coordinate shapes, distances, and numerical relationships between spaces and the structures with which she plans to fill those spaces. To be successful, the architect must apply geometry and calculus to her designs in order to be able to make sure that the artistic vision will "work" in the real world.

Architects, engineers, and artists have been using math through the centuries in creating some of the world's greatest treasures. The Great Pyramid at Giza (the only remaining member of the Seven Wonders of the World), Leonardo da Vinci's "The Last Supper," and the designs of Buckminster Fuller (inventor of the geodesic dome) all feature complex geometric formulations that required a strong command of mathematics in addition to an eye for form and function.

The Gateway Arch in Saint Louis, Missouri, is close to being an inverted catenary (see Section 9.5). It was designed by architect Eero Saarinen. You can learn more about the history of the catenary (the term was first used by mathematician Christiaan Huygens) by exploring the many Internet sites devoted to mathematics and its history, like mathworld.wolfram.com.

Here is a great project idea (encourage your professor to assign it!) Research the mathematics behind the Gateway Arch. You will need to find a formula for a transformed, inverted catenary curve. Use your calculator to graph the formula and create an image that is as close to the actual monument as possible. By linking your grapher to a computer you can print the calculator screen and create an exhibit by putting the image side by side with a photograph of the arch. Augment the picture with an explanation, including the history, and you will have a fine display for your college library or science center.



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THIS CHAPTER CONTINUES the development of nonalgebraic ("transcendental") functions begun in Chapter 8. In the first half we discuss the *inverse trigonometric functions*, singling out three that are important for purposes of integration. Then we turn to certain combinations of exponentials called *hyperbolic functions*, which are remarkably analogous to the familiar trigonometric functions (and easier to discuss in some respects). They are also important in applications. Finally we derive logarithmic formulas for the *inverse hyperbolic functions*, which lead to integration formulas like those involving the inverse trigonometric functions. At that point you will have a substantial list of "standard forms" to take into the next chapter (which is devoted to techniques of integration). More important (in the long run), you will have learned all the "elementary functions of analysis," which are basic working tools of mathematics and its applications.

9.1 Inverse Trigonometric Functions

We began Chapter 8 by seeking a function that would serve as an answer to the antidifferentiation problem

$$\int \frac{dx}{x} = ?$$

Since there seemed to be no other way to proceed, we simply gave a name to the function

$$F(x) = \int_1^x \frac{dt}{t}$$

(the natural logarithm) and then used it to solve our problem by writing

$$\int \frac{dx}{x} = \ln x + C \qquad (x > 0)$$

As we pointed out at the time, *any* missing antiderivative, say

$$\int f(x)\,dx = ?$$

can be supplied in this way, by defining

$$F(x) = \int_a^x f(t)\,dt$$

and observing (by the Fundamental Theorem of Calculus) that $F'(x) = f(x)$.

Ordinarily this is not a profitable thing to do. But as you have seen in Chapter 8 the natural logarithm (and its exponential inverse) have many useful properties that justify our singling them out.

In this section we are going to introduce functions that supply other important missing antiderivatives. One of them, for example, is an answer to the problem

$$\int \frac{dx}{1 + x^2} = ?$$

We could proceed as in the case of the natural logarithm by writing

$$F(x) = \int_0^x \frac{dt}{1 + t^2}$$

Then $F'(x) = 1/(1 + x^2)$ and our problem is (theoretically) solved. Give F a name, tabulate its values, study its properties (including the question of what its inverse is like), and soon it would become a familiar function in much the same way the logarithm has been added to our repertoire. (See Additional Problems 55 and 56 at the end of Chapter 8.)

The reason we do not take this route is that it is unnecessary. For it happens that F is the inverse of a function that is already adequately defined and well known, namely the tangent. It is therefore more natural (although not any easier from a theoretical point of view) to begin with the tangent and then introduce F as its inverse.

These remarks should help you understand why we now investigate what the inverse trigonometric functions are like. It is not because we have suffered an attack of renewed interest in trigonometry, but because of the important role these functions play in calculus.

We begin with the inverse sine. It may seem perverse (after this preamble) to point out that the sine does not even have an inverse! For its graph fails the horizontal line test; given a number y in the range of $y = \sin x$, there are infinitely many values of x in its domain such that $\sin x = y$. (See Figure 1.)

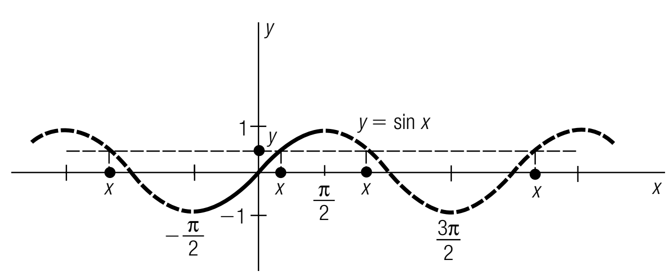

Figure 1 Failure of the Sine to Meet the Horizontal Line Test

This is no problem, however. As in the case of other functions without an inverse, we simply restrict the domain in such a way that $\sin x$ takes each value in its range exactly once. Figure 1 shows that the most natural choice is the domain $[-\pi/2, \pi/2]$. The new sine function (the solid portion of the graph) does have an inverse, namely

$$x = \sin^{-1} y \qquad \text{defined by } y = \sin x,\ -\pi/2 \le x \le \pi/2$$

As usual when dealing with an inverse function, we interchange x and y in order to discuss the new function with its variables labeled conventionally. Hence our formal definition of the inverse sine is as follows.

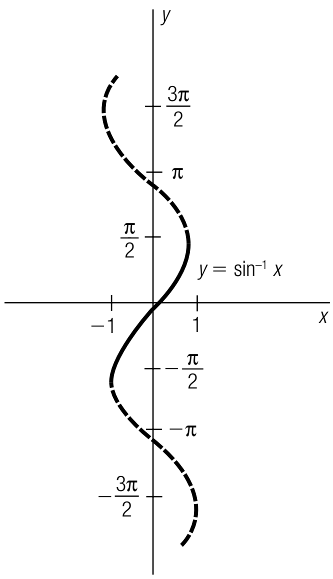

Figure 2 Graph of the Inverse Sine

The **inverse sine** function is given by

$$y = \sin^{-1} x \Leftrightarrow x = \sin y \qquad -\pi/2 \leq y \leq \pi/2$$

It is defined for $-1 \leq x \leq 1$, while its range (the domain of the restricted sine) is $[-\pi/2, \pi/2]$.

The graph of the inverse sine (the reflection of the restricted sine in the line $y = x$) is shown in Figure 2.

Remark

Some books use the notation arcsin x in place of $\sin^{-1} x$. The idea is that $y = \arcsin x$ may be read "y is the arc whose sine is x," that is, $\sin y = x$. (This makes sense in view of the unit circle definitions of the trigonometric functions, where the input is often interpreted as an arc of the circle.) There is no harm in reading $y = \sin^{-1} x$ as "y is the angle whose sine is x," provided that you understand what angle is meant (and that it must be measured in radians to match the numerical output of the inverse sine). The notation and its verbal translation are not important; the essential thing is to know what the inverse sine is. Note particularly that it is not the reciprocal of sine, that is, $\sin^{-1} x \neq (\sin x)^{-1}$.

■ Example 1

Find each of the following.

(a) $\sin^{-1} 1$ (b) $\sin^{-1} \frac{1}{2}$ (c) $\sin^{-1} (-\sqrt{3}/2)$
(d) $\sin^{-1} 0.8$ (e) $\sin^{-1} 2$

Solution

(a) The equation $y = \sin^{-1} 1$ is equivalent to

$$\sin y = 1 \qquad -\pi/2 \leq y \leq \pi/2$$

The only number in $[-\pi/2, \pi/2]$ whose sine is 1 is $y = \pi/2$, so $\sin^{-1} 1 = \pi/2$.

(b) $\sin^{-1} \frac{1}{2} = \pi/6$, because $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ is between $-\pi/2$ and $\pi/2$.

(c) $\sin^{-1}(-\sqrt{3}/2) = -\pi/3$, because $\sin(-\pi/3) = -\sqrt{3}/2$ and $-\pi/3$ is between $-\pi/2$ and $\pi/2$. Note that it is incorrect to write

$$\sin^{-1}(-\sqrt{3}/2) = 4\pi/3$$

as you might be tempted to do because of your experience in trigonometry. For although it is true that $\sin 4\pi/3 = -\sqrt{3}/2$, the number $4\pi/3$ is not in the range of the inverse sine. Worse yet, do not write

$$\sin^{-1}(-\sqrt{3}/2) = 240^\circ \text{ (or even } -60^\circ\text{)}$$

The inverse sine (like the sine) is a function with numerical inputs and outputs; angles (except in radian measure) only muddy the water.

(d) To find $y = \sin^{-1} 0.8$ (equivalent to $\sin y = 0.8$, $-\pi/2 \leq y \leq \pi/2$), we need a table or a calculator. The latter is simplest, for it is programmed to give a numerical answer directly from the inverse sine key. (Be sure your graphing calculator is in radian mode.) Thus $\sin^{-1} 0.8 = 0.92729\ldots$.

(e) The equation $y = \sin^{-1} 2$ is equivalent to

$$\sin y = 2 \qquad -\pi/2 \le y \le \pi/2$$

Since the range of sine is $[-1, 1]$, no such y exists; $\sin^{-1} 2$ is undefined. ■

■ Example 2

Discuss the distinction between the functions

$$f(x) = \sin(\sin^{-1} x) \qquad \text{and} \qquad g(x) = \sin^{-1}(\sin x)$$

Solution

Since the sine and inverse sine are inverse functions, we know that $\sin(\sin^{-1} x) = x$ for all x in the domain of $\sin^{-1}$. (See Section 8.2.) This domain is the closed interval $[-1, 1]$, so the graph of f is as shown in Figure 3 (the solid part of the line $y = x$).

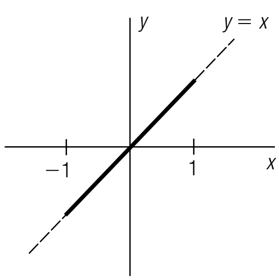

Figure 3 Graph of $f(x) = \sin(\sin^{-1} x)$

The function $g(x) = \sin^{-1}(\sin x)$, on the other hand, is defined for all x. (Why?) It is easy to make the mistake of writing $\sin^{-1}(\sin x) = x$ for all x, in which case the graph of g would be the line $y = x$. The identity holds, however, only in the domain of the restricted sine, that is,

$$\sin^{-1}(\sin x) = x \qquad \text{for } -\pi/2 \le x \le \pi/2$$

When x is outside this domain, things are not so simple. For example,

$$\sin^{-1}(\sin \pi) = \sin^{-1} 0 = 0 \qquad (\text{not } \pi)$$

You should be able to figure out that

$$\begin{array}{ll} \sin^{-1}(\sin x) = x & \text{if } -\pi/2 \le x \le \pi/2 \\ \sin^{-1}(\sin x) = \pi - x & \text{if } \pi/2 \le x \le 3\pi/2 \\ \sin^{-1}(\sin x) = x - 2\pi & \text{if } 3\pi/2 \le x \le 5\pi/2 \end{array}$$

and so on. The graph is shown in Figure 4.

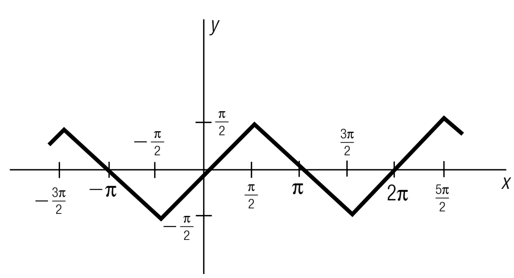

Figure 4 Graph of $y = \sin^{-1}(\sin x)$ ■

■ Example 3

Find the derivative of $y = \sin^{-1} x$.

Solution

Differentiate implicitly in the equivalent equation

$$\sin y = x \qquad -\pi/2 \le y \le \pi/2$$

to obtain

$$\frac{d}{dx}(\sin y) = 1$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

To express this result in terms of x (remembering that $x = \sin y$), we need a relation between $\sin y$ and $\cos y$. It is not hard to dig one up:

$$\sin^2 y + \cos^2 y = 1$$

$$\cos^2 y = 1 - \sin^2 y = 1 - x^2$$

$$\cos y = \pm\sqrt{1 - x^2}$$

The ambiguous sign can be settled by observing that $\cos y \ge 0$ when $-\pi/2 \le y \le \pi/2$. Hence we conclude that

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

As usual, this formula should be built into the Chain Rule:

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}$$

Thus (for example)

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1 - (x^2)^2}}\frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1 - x^4}}$$ ■

Now we turn to the inverse tangent, which is defined as follows. (See Figure 5.)

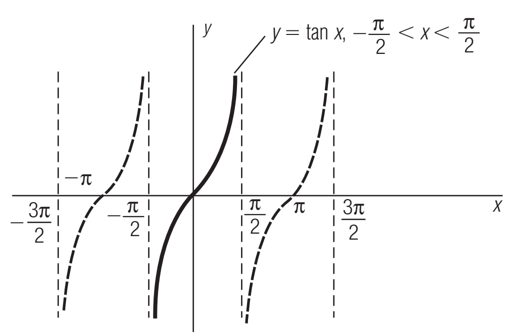

Figure 5 Graph of the Restricted Tangent

The **inverse tangent** function is given by

$$y = \tan^{-1} x \Leftrightarrow x = \tan y \qquad -\pi/2 < y < \pi/2$$

It is defined for all x (because the range of tangent is $\mathcal{R}$), while its range is the domain of the restricted tangent, namely $(-\pi/2, \pi/2)$.

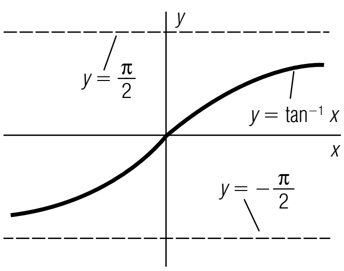

Figure 6 Graph of the Inverse Tangent

The graph of the inverse tangent (the reflection of the restricted tangent in the line $y = x$) is shown in Figure 6.

■ Example 4

Find the derivative of the inverse tangent.

Solution

If $y = \tan^{-1} x$, implicit differentiation in $\tan y = x$ gives

$$\sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

$$D_x \tan^{-1} x = \frac{1}{1 + x^2}$$

Using this formula with the Chain Rule, we find (for example)

$$\frac{d}{dx}(\tan^{-1} e^{-x}) = \frac{1}{1 + (e^{-x})^2} \frac{d}{dx}(e^{-x})$$

$$= \frac{-e^{-x}}{1 + e^{-2x}} = \frac{-e^{x}}{e^{2x} + 1}$$

■

When we come to the inverse secant, the domain to be chosen is not so apparent as in the preceding cases. Look at Figure 7 to see why. There is no single interval in which secant takes on its values exactly once; no matter how we do it, our domain is going to be in two pieces. For the moment, let's postpone a decision.

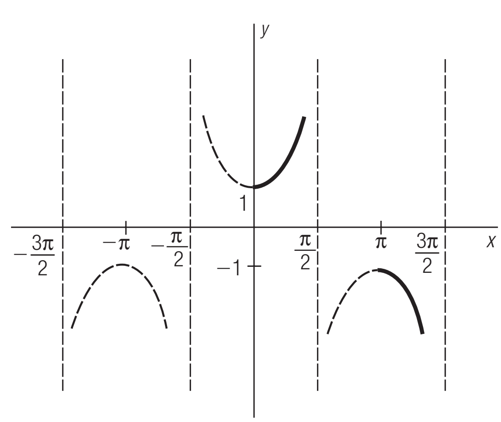

Figure 7 Graph of the Restricted Secant

■ Example 5

Assuming that some decision has been made about the restricted secant, discuss the derivative of its inverse.

Solution

The equation $y = \sec^{-1} x$ is equivalent to $\sec y = x$ (where y lies in the domain not yet specified). Differentiating implicitly, we find

$$\sec y \tan y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x \tan y} \qquad (\text{because } \sec y = x)$$

To express this result entirely in terms of x, we use the identity $\sec^2 y - \tan^2 y = 1$:

$$\tan^2 y = \sec^2 y - 1 = x^2 - 1$$
$$\tan y = \pm\sqrt{x^2 - 1}$$

The ambiguous sign cannot be settled (like it was in Example 3) until we know the domain of the restricted secant. *We choose the domain to make the tangent nonnegative.* Certainly the interval $[0,\pi/2)$ should be part of it; the other part should be an interval in which secant takes its remaining values and tangent is never negative. We select $[\pi,3\pi/2)$.

The **inverse secant** function is given by

$$y = \sec^{-1} x \Leftrightarrow x = \sec y \qquad 0 \le y < \pi/2 \text{ or } \pi \le y < 3\pi/2$$

It is defined for $x \ge 1$ or $x \le -1$ and its range is the union of the intervals $[0,\pi/2)$ and $[\pi,3\pi/2)$.

With this definition in hand, we can finish Example 5 by writing

$$D_x \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$$

■

It may seem underhanded to fix things up (in Example 5) after the fact. There is nothing illegal, immoral, or fattening about it, however, since it is a matter of definition. We could have given the definition first (and then settled the ambiguous sign as in Example 3), but it is better to offer a reason for the definition finally adopted.

Remark

You should be aware that the domain of the restricted secant is not always chosen this way. Sometimes the interval $[\pi,3\pi/2)$ is replaced by $[-\pi,-\pi/2)$ and sometimes by $(\pi/2,\pi]$. The former choice leads to the same derivative as above (why?), but the latter yields

$$D_x \sec^{-1} x = \frac{1}{|x| \sqrt{x^2 - 1}}$$

The absolute value is awkward (and leads to an ambiguous integration formula later), which is why we have chosen a different domain. You must read other books carefully on this point (to avoid confusion due to alternate definitions). One thing, at least, is common to every book, namely the choice of $(0,\pi/2)$ as part of the domain of each restricted trigonometric function f. Hence the evaluation of $f^{-1}(x)$ for $x > 0$ (the situation most often encountered in applications) is no problem.

The only remaining question (which you may have been wondering about) is what happened to $\cos^{-1}$, $\cot^{-1}$, and $\csc^{-1}$? The reason we have left them for last is that in calculus they are superfluous. To see why, consider (for example) the inverse cosine. Figure 8 shows that a good definition is

$$y = \cos^{-1} x \Leftrightarrow x = \cos y \qquad 0 \le y \le \pi$$

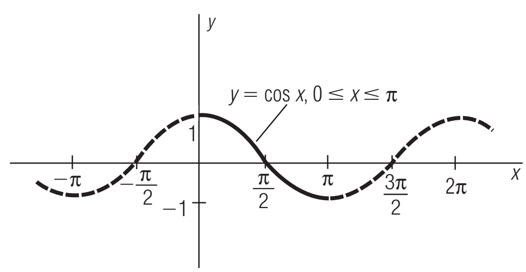

Figure 8 Graph of the Restricted Cosine

By imitating Example 3, you should be able to prove that

$$D_x \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$$

Since this is the negative of the derivative of $\sin^{-1} x$, it is of no interest in antidifferentiation. (See Section 9.2.) Moreover, the inverse cosine can be written in terms of the inverse sine, as the following example shows.

■ Example 6

Explain why $\cos^{-1} x = \pi/2 - \sin^{-1} x$, $-1 \le x \le 1$.

Solution

We know that $\sin^{-1} x$ and $-\cos^{-1} x$ have the same derivative in the open interval $(-1,1)$. Hence they differ by a constant:

$$\sin^{-1} x - (-\cos^{-1} x) = C \qquad (\text{that is, } \sin^{-1} x + \cos^{-1} x = C)$$

Put $x = 0$ in this identity to find $C = \pi/2$. It follows that

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \qquad \text{for } -1 < x < 1$$

To extend the formula to the endpoints of the interval, we check $x = \pm 1$ directly:

$$\sin^{-1} 1 = \frac{\pi}{2} \qquad \text{and} \qquad \cos^{-1} 1 = 0$$

so $\cos^{-1} 1 = \pi/2 - \sin^{-1} 1$; and

$$\sin^{-1}(-1) = -\frac{\pi}{2} \qquad \text{and} \qquad \cos^{-1}(-1) = \pi$$

so $\cos^{-1}(-1) = \pi/2 - \sin^{-1}(-1)$. ■

Remark

The inverse cosine, while superfluous in calculus, is used to find the angle between two vectors. (See Section 15.1.) Hence it is worth remembering.

The inverse cotangent is defined by

$$y = \cot^{-1} x \Leftrightarrow x = \cot y \qquad 0 < y < \pi$$

(Draw the graph of cotangent to see why.) Its derivative is

$$D_x \cot^{-1} x = \frac{-1}{1 + x^2}$$

which is the negative of the derivative of $\tan^{-1} x$. Moreover (as in Example 6), it can be shown that

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x \qquad \text{for all } x$$

It might help your understanding of this section to work through the reasons for these statements.

The inverse cosecant is hardly worth mentioning. For the record, however, we define it by

$$y = \csc^{-1} x \Leftrightarrow x = \csc y \qquad 0 < y \leq \pi/2 \text{ or } \pi < y \leq 3\pi/2$$

Its derivative is

$$D_x \csc^{-1} x = \frac{-1}{x\sqrt{x^2 - 1}}$$

which is the negative of the derivative of $\sec^{-1} x$.

Problem Set 9.1

Find each of the following in exact form (no approximations).

1. $\sin^{-1} \frac{1}{2}$
2. $\cos^{-1} (-1)$
3. $\tan^{-1} \sqrt{3}$
4. $\cot^{-1} 0$
5. $\sec^{-1} 1$
6. $\cos^{-1} 0$
7. $\cos^{-1} 2$
8. $\sin^{-1} (-\frac{1}{2})$
9. $\cot^{-1} (-\sqrt{3})$
10. $\sec^{-1} \sqrt{2}$
11. $\tan^{-1} (-1/\sqrt{3})$
12. $\sec^{-1} \frac{1}{2}$
13. $\tan (\tan^{-1} 2)$
14. $\cos^{-1} (\cos 3\pi/2)$
15. $\cos (\sec^{-1} 3)$
16. $\sin (2 \tan^{-1} 3)$
    Hint: Let $t = \tan^{-1} 3$ and use a formula for $\sin 2t$.
17. $\cos (\sin^{-1} \frac{3}{5} + \cos^{-1} \frac{5}{13})$
    Hint: Use a formula for $\cos (u + v)$.
18. $\tan (\frac{1}{2} \sin^{-1} \frac{3}{5})$

Find the derivative of each of the following functions.

19. $y = \sin^{-1} (x/2)$
20. $y = \tan^{-1} 2x$
21. $y = \cos^{-1} (1/x^2)$
22. $y = \sec^{-1} x^2$
23. $y = \cot^{-1} (1 - x)$
24. $y = \tan^{-1} \sqrt{x}$
25. $y = \sin^{-1} x + \sqrt{1 - x^2}$
26. $y = x^2 \tan^{-1} x$
27. $y = x \sin^{-1} x + \sqrt{1 - x^2}$
28. $y = \sin^{-1} x + x\sqrt{1 - x^2}$
29. $y = \tan^{-1} \left(\dfrac{x - 1}{x + 1}\right)$
30. $y = \sin^{-1} \left(\dfrac{1 - x}{1 + x}\right)$
31. $y = \cot^{-1} (\tan x)$
32. $y = \sec^{-1} (\csc x)$
33. $y = \sin^{-1} (\cos x)$
34. $y = \cos^{-1} (\sin x)$
35. $y = \tan^{-1} x + \frac{1}{2} \ln (1 + x^2)$
36. $y = x \tan^{-1} x - \frac{1}{2} \ln (1 + x^2)$
37. Find dy/dx from the relation
$$\tan^{-1} \frac{y}{x} = \ln \sqrt{x^2 + y^2}$$
38. Some students expect the inverse trigonometric functions to satisfy identities analogous to familiar trigonometric formulas, for example,
$$\tan^{-1} x = \frac{\sin^{-1} x}{\cos^{-1} x}$$
    Give a numerical example showing this formula to be false.
39. Give a numerical example disproving the formula $\sin^{-1} x = (\sin x)^{-1}$.
40. The formula $\cos^{-1} x = \pi/2 - \sin^{-1} x$ $(-1 \le x \le 1)$ was derived in Example 6 by using calculus. Prove it directly from the definitions, as follows.
    (a) Let $y = \sin^{-1} x$. Explain why $x = \cos (\pi/2 - y)$. *Hint*: Recall the cofunction identities from trigonometry.
    (b) In view of the fact that $-\pi/2 \le y \le \pi/2$, explain why the equation $x = \cos (\pi/2 - y)$ is equivalent to $\cos^{-1} x = \pi/2 - y$.
41. Show that $\cot^{-1} x = \pi/2 - \tan^{-1} x$ for all x.
42. Explain why the graphs of $y = \sin (\sin^{-1} x)$ and $y = \cos (\cos^{-1} x)$ are identical segments of the line $y = x$.
43. Confirm that
$$D_x \sin^{-1} (\sin x) = \frac{\cos x}{|\cos x|}$$
    and use the result to check the graph of $y = \sin^{-1} (\sin x)$ in Figure 4.
44. Confirm that
$$D_x \cos^{-1} (\cos x) = \frac{\sin x}{|\sin x|}$$
    and use the result to help sketch the graph of $y = \cos^{-1} (\cos x)$.
45. Confirm that $D_x \tan^{-1} (\cot x) = -1$ and use the result to help sketch the graph of $y = \tan^{-1} (\cot x)$. Watch the domain!
46. Sketch the graph of the inverse cosine.
47. Sketch the graph of the cotangent and explain why it is natural to restrict its domain to $(0, \pi)$ in order to guarantee an inverse.
48. Sketch the graph of the inverse cotangent.
49. Use the formula
$$\frac{dy}{dx} = \frac{1}{dx/dy} \qquad \text{(Section 8.2)}$$
    to find the derivative of $y = \sin^{-1} x$, as follows.
    (a) Explain why
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos (\sin^{-1} x)}$$
    (b) Show that $\cos (\sin^{-1} x) = \sqrt{1 - x^2}$
50. Use the method of Problem 49 to find the derivative of $y = \tan^{-1} x$.

51. Use differentiation to show that

$$\sin^{-1} \frac{x}{\sqrt{1+x^2}} = \tan^{-1} x \qquad \text{for all } x$$

52. Use differentiation to show that

$$\tan^{-1} \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x \qquad \text{if } -1 < x < 1$$

53. Verify the formula

$$\int \sqrt{1-x^2}\, dx = \tfrac{1}{2} x \sqrt{1-x^2} + \tfrac{1}{2} \sin^{-1} x + C$$

54. Derive the formula $D_x \cos^{-1} x = -1/\sqrt{1-x^2}$.

55. Derive the formula $D_x \cot^{-1} x = -1/(1+x^2)$.

56. Since tangent and cotangent are reciprocals, it seems reasonable to expect that $\cot^{-1}(1/x) = \tan^{-1} x$. Investigate the validity of this formula as follows.

(a) Confirm that $D_x \cot^{-1}(1/x) = D_x \tan^{-1} x$ for $x \neq 0$.

(b) It would seem to follow that $\cot^{-1}(1/x) = \tan^{-1} x + C$. Put $x = 1$ to obtain $C = 0$, which apparently establishes the desired formula.

(c) Put $x = -1$ in the formula $\cot^{-1}(1/x) = \tan^{-1} x$ to show that it is false. (!)

(d) Look up Theorem 2, Section 4.4, to discover what went wrong. When is the formula $\cot^{-1}(1/x) = \tan^{-1} x$ correct?

(e) Explain why $\cot^{-1}(1/x) = \pi + \tan^{-1} x$ if $x < 0$.

57. Use differentiation to prove that

$$\sec^{-1} \frac{1}{x} = \begin{cases} \cos^{-1} x & \text{if } 0 < x \le 1 \\ 2\pi - \cos^{-1} x & \text{if } -1 \le x < 0 \end{cases}$$

58. The function $y = \cos^{-1} x - \sec^{-1} x$ has the same value for all x in its domain. But its derivative, far from being zero, does not exist for any value of x. Explain.

59. We suggested in Problems 55 and 56 at the end of Chapter 8 that the inverse sine and inverse tangent could have been defined as integrals,

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}} \qquad \text{and} \quad \tan^{-1} x = \int_0^x \frac{dt}{1+t^2}$$

Having given different definitions, however, we must regard these formulas as unproved. Why are they true?

9.2 Integration Involving Inverse Trigonometric Functions

The most important fact about the inverse trigonometric functions is that they supply powerful new integration techniques. For example, we know that

$$D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

from which it follows that

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

A slightly broader version of this formula is more useful (and is the one you should learn):

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + C$$

(In all integration formulas of this type, involving a constant a^2, we assume that $a > 0$. Otherwise it is sometimes necessary to write $|a|$, which is annoying.)

The above formula can be confirmed by differentiation, but only if the answer is known in advance. To prove it directly, write

$$\sqrt{a^2-x^2} = \sqrt{a^2(1-x^2/a^2)} = a\sqrt{1-x^2/a^2} \qquad (\text{because } a > 0)$$

Hence

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dx}{a\sqrt{1 - x^2/a^2}} = \int \frac{du}{\sqrt{1 - u^2}} \qquad \left[u = \frac{x}{a}, du = \frac{1}{a}dx\right]$$

$$= \sin^{-1} u + C = \sin^{-1} \frac{x}{a} + C$$

■ **Example 1**

Compute the value of $\displaystyle\int_0^1 \frac{dx}{\sqrt{9 - 4x^2}}$.

Solution

Let $u = 2x$, $du = 2\,dx$. Then

$$\int_0^1 \frac{dx}{\sqrt{9 - 4x^2}} = \frac{1}{2}\int_0^2 \frac{du}{\sqrt{9 - u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{3}\Bigg|_0^2$$

$$= \frac{1}{2}\left(\sin^{-1}\frac{2}{3} - \sin^{-1} 0\right)$$

$$= \frac{1}{2}\sin^{-1}\frac{2}{3} \approx 0.365 \qquad \text{(from a calculator)}$$ ■

No worthwhile integration formula is associated with the inverse cosine. For although the equation

$$D_x \cos^{-1} x = \frac{-1}{\sqrt{1 - x^2}}$$

implies that

$$\int \frac{dx}{\sqrt{1 - x^2}} = -\cos^{-1} x$$

we already know that

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$$

(with an arbitrary constant added in each case). Hence we only complicate life by adding a new formula to our table of integrals. The derivative of the inverse tangent, on the other hand,

$$D_x \tan^{-1} x = \frac{1}{1 + x^2}$$

yields the formula

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x + C$$

More generally (as you may confirm)

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Example 2

Find the area under the curve $y = 1/(x^2 + 4)$, $-2 \le x \le 2$.

Solution

The area is

$$\int_{-2}^{2} \frac{dx}{x^2 + 4} = 2\int_{0}^{2} \frac{dx}{x^2 + 4} \qquad \text{(why?)}$$

$$= 2 \cdot \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_0^2 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

The inverse cotangent, like the inverse cosine, is not useful for integration (because its derivative is merely the negative of the derivative of the inverse tangent). Hence we turn to the inverse secant, with derivative

$$D_x \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$$

The corresponding integral formula is

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C$$

or (more generally)

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{x}{a} + C$$

Example 3

Evaluate $\displaystyle\int_1^2 \frac{dx}{x\sqrt{16x^2 - 5}}$.

Solution

Let $u = 4x$, $du = 4\,dx$. Then

$$\int_1^2 \frac{dx}{x\sqrt{16x^2 - 5}} = \int_1^2 \frac{4\,dx}{4x\sqrt{16x^2 - 5}} = \int_4^8 \frac{du}{u\sqrt{u^2 - 5}} = \frac{1}{\sqrt{5}} \sec^{-1} \frac{u}{\sqrt{5}} \Big|_4^8$$

$$= \frac{1}{\sqrt{5}} \left(\sec^{-1} \frac{8}{\sqrt{5}} - \sec^{-1} \frac{4}{\sqrt{5}} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\cos^{-1} \frac{\sqrt{5}}{8} - \cos^{-1} \frac{\sqrt{5}}{4} \right) \qquad \text{(why?)}$$

$$\approx 0.14 \qquad \text{(from a calculator)}$$

Remark

We changed from $\sec^{-1}$ to $\cos^{-1}$ in Example 3 because most calculators do not have an inverse secant key. You should convince yourself that $\sec^{-1}(1/x) = \cos^{-1} x$ when $0 < x \le 1$, while noting that when $-1 \le x < 0$ the formula is false. (See Problem 57, Section 9.1.)

Our last example shows the usefulness of an inverse trigonometric function in an impressive application.

■ Example 4

Hooke's Law says that the restoring force exerted by a spring displaced x units from its natural position is $F = -kx$ ($k > 0$). Newton's Second Law (force equals mass times acceleration) converts this equation to the form $m(d^2x/dt^2) = -kx$, where t is time. Thus the motion of the spring (with displacement x at time t) is described by the second-order differential equation

$$\frac{d^2x}{dt^2} = -a^2x \qquad (a = \sqrt{k/m}\,)$$

Suppose that the motion starts with $x = 0$ and $v = dx/dt = a$ at $t = 0$. (The spring moves from its natural, unstretched position with initial velocity a.) What is the law of motion?

Solution

Reduce the equation to first-order by writing $dv/dt = -a^2x$. The awkward presence of t as the independent variable (when the right side involves x) can be cured by using a device due to Newton:

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} \qquad \text{(by the Chain Rule)}$$

Hence our equation becomes $v\,(dv/dx) = -a^2x$. Separate the variables by writing $v\,dv = -a^2x\,dx$ and integrate:

$$\frac{v^2}{2} = -\frac{a^2x^2}{2} + C_1$$

Since $v = a$ when $x = 0$, we find $C_1 = a^2/2$ and hence $v^2 = a^2(1 - x^2)$. The spring starts out with a positive velocity (which persists during an interval after $t = 0$), so we choose the positive square root when solving for v:

$$v = a\sqrt{1 - x^2}, \text{ that is, } \frac{dx}{dt} = a\sqrt{1 - x^2}$$

Now separate the variables again and integrate:

$$\frac{dx}{\sqrt{1 - x^2}} = a\,dt$$

$$\int \frac{dx}{\sqrt{1 - x^2}} = \int a\,dt$$

$$\sin^{-1} x = at + C_2$$

Since $x = 0$ when $t = 0$, we find $C_2 = 0$ and hence

$$\begin{aligned} \sin^{-1} x &= at \\ x &= \sin at \end{aligned}$$

This is the law of motion we mentioned in Section 7.7 (with the remark that it is not easy to explain). Note the role of the inverse sine in finding it. ■

Remark

In Chapter 20 we will explain why the general solution of the differential equation

$$\frac{d^2x}{dt^2} + a^2x = 0$$

is $x = A\cos at + B\sin at$. (See Problem 43, Section 7.7, where the same statement is made.) If we assume that fact, Example 4 requires considerably less effort. For then we can differentiate the general solution to obtain

$$\frac{dx}{dt} = -aA\sin at + aB\cos at$$

and the initial conditions $x = 0$ and $dx/dt = a$ at $t = 0$ yield

$$0 = A\cos 0 + B\sin 0 \qquad \text{and} \qquad a = -aA\sin 0 + aB\cos 0$$

It follows that $A = 0$ and $B = 1$, so the solution of Example 4 is (as before) $x = \sin at$.

Problem Set 9.2

Find a formula for each of the following.

1. $\displaystyle\int \frac{dx}{\sqrt{25 - x^2}}$

2. $\displaystyle\int \frac{dx}{\sqrt{1 - 9x^2}}$

3. $\displaystyle\int \frac{x\,dx}{\sqrt{9 - x^4}}$ *Hint*: Let $u = x^2$.

4. $\displaystyle\int \frac{dx}{x^2 + 12}$

5. $\displaystyle\int \frac{dx}{9 + 16x^2}$

6. $\displaystyle\int \frac{e^x\,dx}{1 + e^{2x}}$

7. $\displaystyle\int \frac{dx}{x\sqrt{9x^2 - 1}}$

8. $\displaystyle\int \frac{dx}{x\sqrt{16x^2 - 9}}$

9. $\displaystyle\int \frac{dx}{x^2 - 2x + 5}$ *Hint*: Complete the square to write $x^2 - 2x + 5 = (x - 1)^2 + 4$.

10. $\displaystyle\int \frac{dx}{\sqrt{6x - x^2}}$ *Hint*: Complete the square.

11. $\displaystyle\int \frac{5 - 2x}{\sqrt{9 + 8x - x^2}}\,dx$ *Hint*: Let $u = 9 + 8x - x^2$. To fit du, write $5 - 2x = (8 - 2x) - 3$ and break into two integrals.

12. $\displaystyle\int \frac{2x + 3}{x^2 - 4x + 5}\,dx$

Evaluate each of the following.

13. $\displaystyle\int_{-2}^{2} \frac{dx}{\sqrt{16 - x^2}}$

14. $\displaystyle\int_{0}^{1} \frac{x\,dx}{\sqrt{4 - x^4}}$

15. $\displaystyle\int_{0}^{\pi/2} \frac{\sin x\,dx}{\sqrt{4 - \cos^2 x}}$

16. $\displaystyle\int_{1}^{3} \frac{dx}{x^2 + 3}$

17. $\displaystyle\int_{0}^{1} \frac{x + 1}{x^2 + 1}\,dx$ *Hint*: Express the integrand as a sum.

18. $\displaystyle\int_{0}^{\pi/2} \frac{\cos x\,dx}{9 + \sin^2 x}$

19. $\displaystyle\int_{3}^{4} \frac{dx}{x\sqrt{x^2 - 4}}$

20. $\displaystyle\int_2^5 \frac{dx}{x\sqrt{9x^2-16}}$

21. $\displaystyle\int_3^5 \frac{dx}{x^2-6x+13}$

22. $\displaystyle\int_0^1 \frac{dx}{\sqrt{3+2x-x^2}}$

23. $\displaystyle\int_{-1}^1 \frac{x-2}{\sqrt{9-4x^2}}\,dx$

24. $\displaystyle\int_{-2}^2 \frac{x-1}{16x^2+25}\,dx$

25. Find the area of the region bounded by the graphs of $y = 1/\sqrt{1-x^2}$ and $y = 2$.

26. Find the area under the curve $y = 9/(9+x^2)$, $-3 \le x \le 3$.

27. Find the volume of the solid generated by rotating the region under the curve

$$y = \frac{1}{\sqrt{9+x^2}} \qquad 0 \le x \le 3$$

about the x axis.

28. Repeat Problem 27 for rotation about the y axis.

29. Use calculus to find the length of the curve

$$y = \sqrt{r^2-x^2} \qquad -r \le x \le r$$

How could the result have been predicted?

30. Solve the initial value problem $dy/dx = 16 + y^2$, where $y = 0$ when $x = 0$.

31. Solve the initial value problem $dy/dx = \sqrt{9-y^2}$, where $y = 0$ when $x = 1$.

32. Find the general solution of the differential equation $dy/dx = 2xy\sqrt{y^2-4}$. *Hint*: First dispose of the special solutions $y = 0$ and $y = \pm 2$.

33. Find the general solution of the differential equation $dy/dx = (4y^2+1)/x$.

34. The motion of a spring with displacement x at time t is described by $d^2x/dt^2 = -a^2x$. If the spring is released from a stretched position of $x = 1$ at $t = 0$, find its law of motion. *Hint*: "Released" means that the velocity is zero at $t = 0$. Immediately thereafter it is negative (which will tell you which square root to choose when the time comes).

35. In Problem 34 suppose that the initial velocity is a. Find the law of motion.

36. The beacon of a lighthouse 1 km from a straight shore revolves five times per minute. Find the speed of its beam along the shore in two ways, as follows. (See Figure 1.)

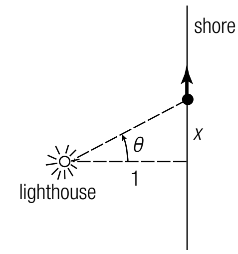

Figure 1 Lighthouse Beam Moving along a Shore

(a) Use the relation $x = \tan\theta$ to show that

$$\frac{dx}{dt} = 600\pi \sec^2\theta \text{ km/hr}$$

(Compare with Problem 36, Section 5.1.)

(b) Use the relation $\theta = \tan^{-1} x$ to show that

$$\frac{dx}{dt} = 600\pi\,(1+x^2) \text{ km/hr}$$

(c) Reconcile the answers. (Note that the use of an inverse trigonometric function offers no advantage, although it is a refreshing change. Most applications involving trigonometry can be treated either way.

37. A camera located 10 meters from the finish line is televising a race. (See Figure 2.) When the runners are 10 meters from the finish line they are going 9 m/sec. How fast is the camera turning at that instant?

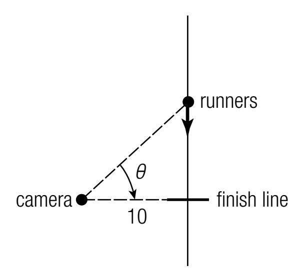

Figure 2 Televising a Race

38. A painting 5 ft high is hung on a wall so that its lower edge is 1 ft above eye level. How far from the wall should an observer stand for the best view? ("Best view" means maximum angle between the lines of sight to the top and bottom of the painting.) *Hint*: If θ is the angle and x is the distance from the wall, then

$$\theta = \cot^{-1}\frac{x}{6} - \cot^{-1} x \qquad \text{(Draw a picture!)}$$

39. A diver is descending vertically from the center of a hemispherical tank (Figure 3) at the rate of 2 ft/sec. A light at the edge of the tank throws her shadow on the curved surface of the tank. How fast is her shadow moving along the tank when she is halfway down?

Hint: First verify that $\alpha = \tan^{-1}(h/r) = \theta/2$. Also note (from trigonometry) that $s = r\theta$.

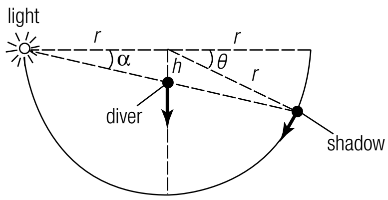

Figure 3 Shadow Moving along a Swimming Tank

40. Derive the formula

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

by differentiating the right side. (Note the role of the assumption $a > 0$.)

41. Derive the formula

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

in two ways, as follows.

(a) Make an appropriate substitution to reduce it to the known formula

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x + C$$

(b) Confirm it by differentiation.

42. Derive the formula

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

in two ways, as follows.

(a) Make an appropriate substitution to reduce it to the known formula

$$\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C$$

(b) Confirm it by differentiation.

Use differentiation to confirm each of the following integration formulas. (Later we will show how to discover them, rather than merely checking them.)

43. $\displaystyle\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1 - x^2} + C$

44. $\displaystyle\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1 + x^2} + C$

45. $\displaystyle\int \sec^{-1} x\,dx = x\sec^{-1} x - \ln|x + \sqrt{x^2 - 1}| + C$

46. $\displaystyle\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

47. In previous chapters we have often computed

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \frac{\pi a^2}{2}$$

by using the area interpretation of integral. Confirm this result by using Problem 46.

48. The integrand of

$$\int_0^2 \frac{dx}{\sqrt{4 - x^2}}$$

is unbounded in the domain of integration, so the integral does not exist. (See Section 6.2.) We can, however, compute

$$\int_0^t \frac{dx}{\sqrt{4 - x^2}}$$

for values of t close to 2 (and less than 2). What is the limit of the result as $t \to 2$? (The original integral may be assigned this value; we will discuss such "improper integrals" later. Note that in this example it would be easy to overlook the difficulty and compute the integral directly in terms of the inverse sine. That does not always work, however.)

49. What does the area under the curve

$$y = \frac{1}{1 + x^2} \qquad 0 \le x \le b$$

approach as b increases without bound?

9.3 Hyperbolic Functions

As you know from trigonometry, the coordinates of any point (x,y) on the unit circle can be written in the form $x = \cos t$, $y = \sin t$, where t is the measure of the arc from $(1,0)$ to (x,y). (See Figure 1.) Since $t = \theta$ (in radians) and since the area of a circular sector of radius r and central angle θ is $\frac{1}{2}r^2\theta$, we may also interpret t as twice the area of the shaded region in Figure 1:

$$2(\text{area of sector}) = 2 \cdot \tfrac{1}{2} \cdot 1^2\theta = t$$

This fact leads directly to the subject of this section. For suppose we try the same idea in connection with the unit hyperbola (the curve $x^2 - y^2 = 1$ in Figure 2). Letting t be twice the area of the shaded region (drawn by analogy to Figure 1), we propose to find x and y in terms of t. The resulting functions are called the *hyperbolic* cosine and sine, respectively; their similarity to the *circular* functions $x = \cos t$ and $y = \sin t$ is striking. Moreover, they turn out to be useful in unexpected ways (which is why they are included in this chapter).

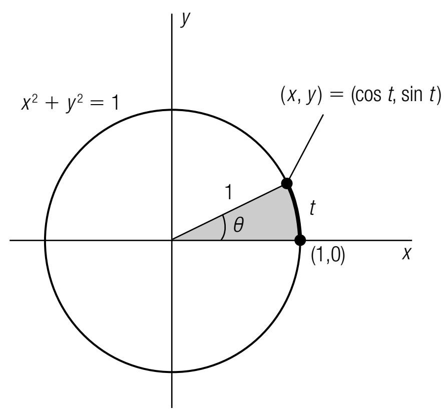

Figure 1 Unit Circle Definition of Sine and Cosine

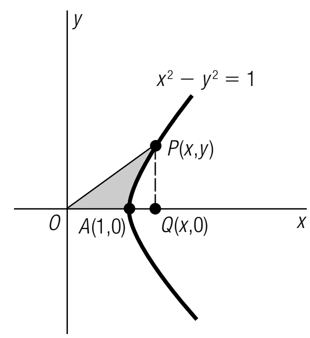

Figure 2 Unit Hyperbola Definition of Sine and Cosine

Anticipating the solution (which is given in an optional note at the end of this section), we assert that

$$x = \tfrac{1}{2}(e^t + e^{-t}) \qquad \text{and} \qquad y = \tfrac{1}{2}(e^t - e^{-t})$$

These remarkable formulas (one wonders if there are similar exponential expressions for $x = \cos t$ and $y = \sin t$!) provide the starting point of our discussion.

The Hyperbolic Functions

The **hyperbolic** sine, cosine, tangent, . . . are defined by

$$\sinh t = \tfrac{1}{2}(e^t - e^{-t}) \qquad \coth t = \frac{\cosh t}{\sinh t} = \frac{e^t + e^{-t}}{e^t - e^{-t}},\ t \neq 0$$

$$\cosh t = \tfrac{1}{2}(e^t + e^{-t}) \qquad \operatorname{sech} t = \frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}}$$

$$\tanh t = \frac{\sinh t}{\cosh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}} \qquad \operatorname{csch} t = \frac{1}{\sinh t} = \frac{2}{e^t - e^{-t}},\ t \neq 0$$

The fundamental identity for hyperbolic functions (like $\cos^2 t + \sin^2 t = 1$ in trigonometry) is

$$\cosh^2 t - \sinh^2 t = 1$$

This follows from the fact that the point $(x,y) = (\cosh t, \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$. Other identities can be derived from this one, such as

$$\tanh^2 t + \operatorname{sech}^2 t = 1 \qquad \text{and} \qquad \coth^2 t - \operatorname{csch}^2 t = 1$$

but they are easily confused with similar formulas in trigonometry; we don't recommend your trying to learn them. The points of similarity and difference between the circular and hyperbolic functions are so unpredictable that you should take nothing for granted. For example, hyperbolic functions are even and odd in the same pattern as the trigonometric functions:

$$\sinh(-t) = -\sinh t \qquad \cosh(-t) = \cosh t \qquad \tanh(-t) = -\tanh t$$

and so on. (Why?) On the other hand, none of the hyperbolic functions is periodic, as you can see from their definition.

■ Example 1

Find the derivative of $y = \sinh x$.

Solution

$$D_x \sinh x = \tfrac{1}{2} D_x(e^x - e^{-x}) - \tfrac{1}{2}(e^x + e^{-x}) \cosh x$$

Compare the simplicity of this argument with the development of $D_x \sin x = \cos x$ in Section 2.5! Evidently life among the hyperbolic functions is going to be easier than trigonometry in some respects. Their derivatives (which are left for the problem set) are given below. ■

$D_x \sinh x = \cosh x$	$D_x \coth x = -\operatorname{csch}^2 x$
$D_x \cosh x = \sinh x$	$D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x$
$D_x \tanh x = \operatorname{sech}^2 x$	$D_x \operatorname{csch} x = -\operatorname{csch} x \coth x$

■ Example 2

Discuss the graph of $y = \sinh x$.

Solution

As noted earlier, sinh is an odd function, so its graph is symmetric about the origin. It is defined for all x, and for large x its values are close to $\frac{1}{2}e^x$. (Why?) Since

$$D_x \sinh x = \cosh x = \tfrac{1}{2}(e^x + e^{-x}) > 0 \qquad \text{for all } x$$

the graph of sinh is always rising. Moreover,

$$D_x^{\,2} \sinh x = \sinh x = \tfrac{1}{2}(e^x - e^{-x}) > 0 \qquad \text{for } x > 0$$

so the graph is concave up in $(0, \infty)$. It passes through the origin because

$$\sinh 0 = \tfrac{1}{2}(e^0 - e^0) = 0$$

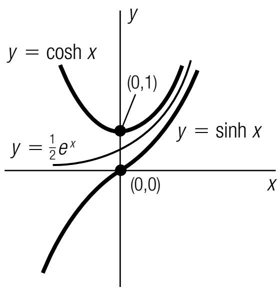

Figure 3 Graphs of sinh and cosh

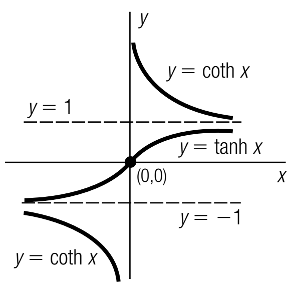

Figure 4 Graphs of tanh and coth

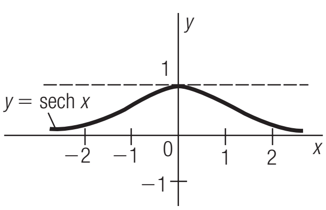

Figure 5 Graph of sech

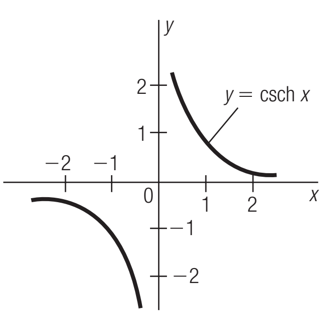

Figure 6 Graph of csch

Hence it has the appearance shown in Figure 3.

We leave it to you to discuss the graph of $y = \cosh x$ (also shown in Figure 3). ■

■ Example 3

Discuss the graph of $y = \tanh x$.

Solution

Since $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$, the domain is $\mathcal{R}$. The graph is symmetric about the origin (because tanh is odd). For large x the values of $\tanh x$ are close to $x \to 1$, while $\tanh x \to -1$ when $x \to -\infty$. (Why?) Hence the lines $y = \pm 1$ are asymptotes. Since

$$D_x \tanh x = \operatorname{sech}^2 x > 0 \qquad \text{for all } x$$

the graph is always rising. Moreover,

$$\begin{aligned} D_x^{\,2} \tanh x &= 2 \operatorname{sech} x(-\operatorname{sech} x \tanh x) \\ &= -2 \operatorname{sech}^2 x \tanh x < 0 \qquad \text{for } x > 0 \end{aligned}$$

Thus the graph is concave down in $(0,\infty)$. It passes through the origin because $\tanh 0 = 0$. (See Figure 4, which also shows the graph of $y = \coth x$.)

The graphs of $y = \operatorname{sech} x$ and $y = \operatorname{csch} x$ (which you should verify along with the graph of $y = \coth x$) are shown in Figures 5 and 6. ■

■ Example 4

In 1700 Jakob Bernoulli proved that a flexible chain or cable hanging from its ends (and supporting no other weight but its own) takes the shape of the curve $y = a \cosh (x/a)$ (called a *catenary* from the Latin word for chain). Find the length of the catenary $y = \cosh x$ between $x = -1$ and $x = 1$. (See the graph of cosh in Figure 3.)

Solution

We use the formula

$$s = \int_{-1}^{1} \sqrt{1 + y'^2}\, dx$$

from Section 7.3. Since $y' = \sinh x$, we find

$$\begin{aligned} \sqrt{1 + y'^2} &= \sqrt{1 + \sinh^2 x} = \sqrt{\cosh^2 x} \qquad \text{(why?)} \\ &= \cosh x \qquad \text{(why?)} \end{aligned}$$

Hence

$$\begin{aligned} s &= \int_{-1}^{1} \cosh x\, dx = 2\int_{0}^{1} \cosh x\, dx \qquad \text{(why?)} \\ &= 2 \sinh x \Big|_0^1 \qquad \text{(why?)} \\ &= 2(\sinh 1 - \sinh 0) = 2 \sinh 1 \qquad (\text{because } \sinh 0 = 0) \end{aligned}$$

Some calculators have a sinh x key, and give $\sinh 1 \approx 1.1752$ directly. We can also write

$$\sinh 1 = \tfrac{1}{2}(e - e^{-1}) \approx 1.1752$$

In any case the length of the curve is approximately $s \approx 2.35$. ■

■ Example 5

Compute

$$\int_0^{\ln 2} \sinh t \cosh^2 t \, dt$$

Solution

Make the substitution $u = \cosh t$. Then $du = \sinh t \, dt$ and the integral takes the form $\int u^2 \, du$. The new limits correspond to $t = 0$ and $t = \ln 2$, respectively:

$$t = 0,\ u = \cosh 0 = 1$$
$$t = \ln 2,\ u = \cosh(\ln 2) = \tfrac{1}{2}(e^{\ln 2} + e^{-\ln 2}) = \tfrac{1}{2}(2 + \tfrac{1}{2}) = \tfrac{5}{4}$$

Hence

$$\int_0^{\ln 2} \sinh t \cosh^2 t \, dt = \int_1^{5/4} u^2 \, du = \frac{61}{192}$$ ■

Optional Note (on finding $x = \cosh t$ and $y = \sinh t$ in terms of t)

In Figure 2 we show a point $P(x,y)$ on the unit hyperbola $x^2 - y^2 = 1$. Since t is twice the area shaded in this figure, we have

$$\frac{t}{2} = (\text{area of triangle } OPQ) - (\text{area of region } APQ)$$

The region APQ is bounded by the hyperbola, the x axis, and the vertical lines through A and Q, so we find it by integration:

$$\text{area of region } APQ = \int_1^x \sqrt{u^2 - 1} \, du$$

(We use u in the integrand to avoid confusion with x in the upper limit.) The area of triangle OPQ is of course

$$\tfrac{1}{2}(\text{base})(\text{altitude}) = \tfrac{1}{2}xy = \tfrac{1}{2}x\sqrt{x^2 - 1}$$

Hence

$$t = x\sqrt{x^2 - 1} - 2\int_1^x \sqrt{u^2 - 1} \, du$$

This looks hopeless to solve for x in terms of t. We can get rid of the integral, however, by differentiation (using the Fundamental Theorem of Calculus):

$$\frac{dt}{dx} = x \cdot \frac{x}{\sqrt{x^2 - 1}} + \sqrt{x^2 - 1} - 2\sqrt{x^2 - 1} = \frac{1}{\sqrt{x^2 - 1}}$$

Now separate the variables and integrate:

$$\frac{dx}{\sqrt{x^2 - 1}} = dt$$

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \int dt \tag{1}$$

It may appear that we are going in circles (first differentiating and then integrating). The integral on the left, however, may be found in Additional Problem 27 at the end of Chapter 8:

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + C$$

Of course we do not expect you to know this formula; not surprisingly, we will derive it by using hyperbolic functions (in the next section). Meanwhile, simply accept it as one that can be checked by differentiation (independently of hyperbolic functions). Then (1) becomes

$$\ln |x + \sqrt{x^2 - 1}| + C = t$$

Since $x \geq 1$ (see Figure 2), we may drop the absolute value. Moreover, $C = 0$ because $t = 0$ when $x = 1$. (Why?) Hence

$$\ln (x + \sqrt{x^2 - 1}) = t$$

Now it is relatively easy to solve for x:

$$\begin{aligned} x + \sqrt{x^2 - 1} &= e^t \\ \sqrt{x^2 - 1} &= e^t - x \\ x^2 - 1 &= (e^t - x)^2 = e^{2t} - 2xe^t + x^2 \\ 2xe^t &= e^{2t} + 1 \\ x &= \frac{e^{2t} + 1}{2e^t} = \frac{1}{2}(e^t + e^{-t}) \end{aligned}$$

To find y, use the fact that $x^2 - y^2 = 1$:

$$\begin{aligned} y^2 &= x^2 - 1 = \tfrac{1}{4}(e^t + e^{-t})^2 - 1 = \tfrac{1}{4}(e^{2t} + 2 + e^{-2t}) - 1 \\ &= \tfrac{1}{4}(e^{2t} - 2 + e^{-2t}) = \tfrac{1}{4}(e^t - e^{-t})^2 \end{aligned}$$

Since $y \geq 0$, we have

$$y = \tfrac{1}{2}|e^t - e^{-t}| = \tfrac{1}{2}(e^t - e^{-t}) \qquad \text{(because } t \geq 0\text{)}$$

Problem Set 9.3

1. What curve is described by the parametric equations $x = \cosh t$, $y = \sinh t$? As t increases through its domain $(-\infty, \infty)$, how does the point (x, y) move along this curve?
2. Explain why sinh is an odd function.
3. Explain why cosh is an even function.
4. Use Problems 2 and 3 to show that tanh, coth, and csch are odd, while sech is even.
5. Use the formula $\cosh^2 t - \sinh^2 t = 1$ to derive the identity $\tanh^2 t + \operatorname{sech}^2 t = 1$.
6. Use the formula $\cosh^2 t - \sinh^2 t = 1$ to derive the identity $\coth^2 t - \operatorname{csch}^2 t = 1$.
7. Given the value $\tanh t = -\frac{3}{4}$, find the value of each of the remaining hyperbolic functions. Include an explanation of why each value is unique.
8. Repeat Problem 7 given the value $\sinh t = 2$.

Use the definitions of sinh and cosh to derive the following formulas.

9. $\sinh (u + v) = \sinh u \cosh v + \cosh u \sinh v$
10. $\cosh (u + v) = \cosh u \cosh v + \sinh u \sinh v$

Use Problems 9 and 10 to derive the following formulas.

11. $\sinh (u - v) = \sinh u \cosh v - \cosh u \sinh v$
12. $\cosh (u - v) = \cosh u \cosh v - \sinh u \sinh v$

13. $\sinh 2t = 2 \sinh t \cosh t$

14. $\cosh 2t = \cosh^2 t + \sinh^2 t$

Use Problem 14 (together with $\cosh^2 t - \sinh^2 t = 1$) to derive the following formulas.

15. $\sinh^2 t = \frac{1}{2}(\cosh 2t - 1)$

16. $\cosh^2 t = \frac{1}{2}(\cosh 2t + 1)$

17. Use the definition of cosh to prove that

$$D_x \cosh x = \sinh x$$

Use the derivatives of sinh and cosh to prove the following.

18. $D_x \tanh x = \operatorname{sech}^2 x$

19. $D_x \coth x = \operatorname{csch}^2 x$

20. $D_x \operatorname{sech} x = -\operatorname{sech} x \tanh x$

21. $D_x \operatorname{csch} x = -\operatorname{csch} x \coth x$

The graph of each of the following functions is shown in the text. Confirm that it is correct by discussing domain, symmetry, asymptotes, extreme values, and concavity.

22. $y = \cosh x$

23. $y = \coth x$

24. $y = \operatorname{sech} x$

25. $y = \operatorname{csch} x$

26. Show that the function $y = A \cosh x + B \sinh x$ satisfies the differential equation $y'' - y = 0$. (Compare with the fact that $y = A \cos x + B \sin x$ satisfies $y'' + y = 0$.)

Find the derivative of each of the following functions.

27. $y = \sinh 2x$

28. $y = \cosh (1 - x)$

29. $y = \tanh x^2$

30. $y = \sinh t + \cosh t$

30. $y = \sinh t - \cosh t$

32. $y = \cosh^2 t - \sinh^2 t$

33. $y = \sinh x$

34. $y = \dfrac{\tanh x}{x}$

35. $y = \ln \cosh x$

36. $y = \ln \sinh x$

37. $y = e^{-x} \sinh x$

38. $y = e^{2x} \cosh x$

39. For what value of x does the function $y = \sinh x + 2 \cosh x$ have its minimum value?

40. Repeat Problem 39 for the function $y = 3 \cosh x - 2 \sinh x$.

Evaluate each of the following integrals.

41. $\displaystyle\int_0^{\ln 2} \sinh x \, dx$ (The answer is a rational number.)

42. $\displaystyle\int_0^{\ln 3} \cosh 2x \, dx$ (The answer is a rational number.)

43. $\displaystyle\int_0^1 \operatorname{sech}^2 x \, dx$

44. $\displaystyle\int_0^2 \operatorname{sech} \frac{x}{2} \tanh \frac{x}{2} \, dx$

45. $\displaystyle\int_0^1 \tanh x \, dx$ *Hint*: Make the substitution $u = \cosh x$.

46. $\displaystyle\int_1^2 \coth x \, dx$ [Can you reduce the answer to the form $\ln (e^2 + 1) - 1$?]

47. Find the area of the region bounded by the curves $y = \sinh x$ and $y = \cosh x$, the y axis, and the line $x = 1$. (Can you reduce the answer to $1 - e^{-1}$?)

48. Find the area of the region bounded by the curve $y = \tanh x$, the y axis, and the lines $x = 1$ and $y = 1$. (Use the hint in Problem 45.)

49. The region under the curve $y = \tanh x$, $0 \le x \le 1$, is rotated about the x axis. Find the volume of the resulting solid of revolution. *Hint*: Use Problem 5.

50. The region under the curve $y = \cosh x$, $0 \le x \le 1$, is rotated about the x axis. Find the volume of the resulting solid of revolution. *Hint*: Use Problem 16.

51. Find the surface area of the solid of revolution in Problem 50.

52. Find the length of the catenary

$$y = a \cosh (x/a)$$

between $x = -a$ and $x = a$.

53. Show that the length of the catenary

$$y = a \cosh (x/a)$$

from $(0,a)$ to (x,y) is $s = a \sinh (x/a)$.

54. Why is it meaningless to ask for the derivative of $y = \sin^{-1} (\cosh x)$?

55. Euler's formula for imaginary exponents (Problem 49, Section 8.6) says that $e^{ix} = \cos x + i \sin x$.

(a) Explain why $e^{-ix} = \cos x - i \sin x$.

(b) Show that

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

and

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

(These remarkable formulas put the sine and cosine in the context of complex-valued functions. From the point of view of exponentials, the real functions sinh and cosh are simpler.)

56. A formula due to Abraham De Moivre (1667–1754) says that if n is any positive integer, then

$$(\cos t + i \sin t)^n = \cos nt + i \sin nt$$

(a) Use mathematical induction, together with the formulas for $\sin(u + v)$ and $\cos(u + v)$, to prove De Moivre's theorem.

(b) Assuming that complex exponentials obey the usual rules, give a simpler argument based on Euler's formula.

(c) Prove the hyperbolic formula

$$(\cosh t + \sinh t)^n = \cosh nt + \sinh nt$$

noting how uncomplicated it is compared to De Moivre's theorem.

9.4 Inverse Hyperbolic Functions

We saw in Section 9.2 that the inverse trigonometric functions are important for purposes of integration. Because of the analogy between trigonometric ("circular") functions and hyperbolic functions, we expect the same thing to be true of the inverse hyperbolic functions. You may be pleased to learn that these functions are ripe for the picking. Unlike the inverse trigonometric functions they involve nothing essentially new, but can be expressed in terms of the natural logarithm.

To see what we mean, consider the inverse hyperbolic sine. You might expect that we would proceed as so often before, observing from Figure 3 in the last section that $f(x) = \sinh x$ has an inverse because it is continuous and increasing. We define it by writing

$$y = \sinh^{-1} x \Leftrightarrow x = \sinh y$$

While this is reasonable enough, a second thought should occur to us. Since

$$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y})$$

why not find the inverse by solving for y in terms of x? This would make the symbol $\sinh^{-1}$ superfluous (because an explicit formula for it exists).

■ Example 1

Find a formula for $y = \sinh^{-1} x$.

Solution

As suggested above, we solve for y in the equation

$$x = \tfrac{1}{2}(e^y - e^{-y})$$

The algebra is interesting:

$$\begin{aligned} x &= \tfrac{1}{2}(e^y - e^{-y}) \\ 2x &= e^y - e^{-y} \\ 2xe^y &= e^{2y} - 1 \end{aligned}$$

Regarding this equation as quadratic in e^y, $(e^y)^2 - 2xe^y - 1 = 0$, we use the quadratic formula (with $a = 1$, $b = -2x$, $c = -1$):

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Since e^y is always positive, and $\sqrt{x^2 + 1} > x$ for all x, the ambiguous sign must be plus. Hence

$$e^y = x + \sqrt{x^2 + 1}$$
$$y = \ln\,(x + \sqrt{x^2 + 1})$$

In other words, the inverse hyperbolic sine is really just a logarithm:

$$\sinh^{-1} x = \ln\,(x + \sqrt{x^2 + 1})$$

■

■ Example 2

Find the derivative of $y = \sinh^{-1} x$.

Solution

The most direct procedure is to differentiate the above logarithm. It is easier, however, to differentiate implicitly in the equation $\sinh y = x$:

$$\cosh y \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

Since $\cosh^2 y - \sinh^2 y = 1$, we find

$$\cosh y = \sqrt{\sinh^2 y + 1} = \sqrt{x^2 + 1}$$

(Why is the positive square root chosen?) Hence

$$D_x \sinh^{-1} x = \frac{1}{\sqrt{x^2 + 1}}$$

■

The integration formula corresponding to this result is

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \sinh^{-1} x + C$$

or (more generally)

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a} + C$$

(Confirm!) While this is good enough as it stands, it is often more useful to write it as a logarithm (using the formula for $\sinh^{-1}$):

$$\sinh^{-1}\frac{x}{a} = \ln\left[\frac{x}{a} + \sqrt{(x/a)^2 + 1}\right] = \ln\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right)$$
$$= \ln(x + \sqrt{x^2 + a^2}) - \ln a$$

Since $-\ln a$ is a constant, we may drop it in the integration formula, obtaining

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2}) + C$$

The inverse hyperbolic cosine is less useful (because the domain of cosh must be artificially restricted before the inverse can be said to exist). (See Figure 3, Section 9.3.) We define it by

$$y = \cosh^{-1} x \Leftrightarrow x = \cosh y \qquad y \geq 0$$

Proceeding as in Example 1, we find

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \qquad (x \geq 1)$$

and (as in Example 2)

$$D_x \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \qquad (x > 1)$$

This leads to

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} + C \qquad (x > a)$$

or, equivalently,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln(x + \sqrt{x^2 - a^2}) + C \qquad (x > a)$$

Since these formulas involve the restriction $x > a$ (whereas the integrand is defined for $x < -a$ as well as for $x > a$), it is desirable to find an unrestricted formula. This is not hard to do. Simply forget about the inverse hyperbolic cosine (it has served its purpose by leading us to the formula) and use an absolute value sign:

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

This holds in the domain $x < -a$ as well as $x > a$, as you can check.

Example 3

Find the inverse of the hyperbolic tangent.

Solution

Figure 4 in Section 9.3 shows that no restriction on tanh is necessary to guarantee an inverse. Hence

$$y = \tanh^{-1} x \Leftrightarrow x = \tanh y$$

To find the inverse, solve for y:

$$\begin{aligned} x &= \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1} \\ xe^{2y} + x &= e^{2y} - 1 \\ 1 + x &= e^{2y}(1 - x) \\ e^{2y} &= \frac{1 + x}{1 - x} \\ 2y &= \ln \frac{1 + x}{1 - x} \\ y &= \frac{1}{2} \ln \frac{1 + x}{1 - x} \end{aligned}$$

Thus we have derived the formula

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1 + x}{1 - x} \qquad (|x| < 1)$$

■

To differentiate this, write it in the simpler form

$$y = \frac{1}{2} [\ln (1 + x) - \ln (1 - x)]$$

Then

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1 + x} + \frac{1}{1 - x}\right) = \frac{1}{2} \cdot \frac{2}{1 - x^2} = \frac{1}{1 - x^2}$$

and hence

$$D_x \tanh^{-1} x = \frac{1}{1 - x^2} \qquad (|x| < 1)$$

A similar discussion of the inverse hyperbolic cotangent shows that

$$\coth^{-1} x = \frac{1}{2} \ln \frac{x + 1}{x - 1} \qquad (|x| > 1)$$

and

$$D_x \coth^{-1} x = \frac{1}{1 - x^2} \qquad (|x| > 1)$$

Note that $\tanh^{-1} x$ and $\coth^{-1} x$ have the same derivative, but in different domains. There are two corresponding integration formulas:

$$\int \frac{dx}{1 - x^2} = \tanh^{-1} x + C \qquad (|x| < 1)$$

and

$$\int \frac{dx}{1 - x^2} = \coth^{-1} x + C \qquad (|x| > 1)$$

or (more generally)

$$\int \frac{dx}{a^2 - x^2} = \begin{cases} \dfrac{1}{a} \tanh^{-1} \dfrac{x}{a} + C & (|x| < a) \\ \dfrac{1}{a} \coth^{-1} \dfrac{x}{a} + C & (|x| > a) \end{cases}$$

A single version of these formulas is sometimes more useful, and may be obtained from the logarithmic formulas for $\tanh^{-1} x$ and $\coth^{-1} x$ given above:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$$

We will confirm this in the case $|x| > a$ and leave the case $|x| < a$ for you:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \coth^{-1} \frac{x}{a} + C = \frac{1}{a} \cdot \frac{1}{2} \ln \left(\frac{x/a + 1}{x/a - 1} \right) + C$$

$$= \frac{1}{2a} \ln \frac{x + a}{x - a} + C = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$$

$\left(\text{because } \left| \dfrac{a + x}{a - x} \right| = \dfrac{x + a}{x - a} \text{ when } |x| > a\right)$.

The inverse hyperbolic secant and cosecant are relatively unimportant, so we will not dwell on them. They are defined by

$$y = \operatorname{sech}^{-1} x \Leftrightarrow x = \operatorname{sech} y \qquad y \geq 0$$
$$y = \operatorname{csch}^{-1} x \Leftrightarrow x = \operatorname{csch} y$$

(See Figures 5 and 6, Section 9.3, to understand why a restriction is needed in the first case but not the second.) It follows that

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x} \qquad \text{and} \qquad \operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x} \qquad \text{(why?)}$$

which means that both functions can be differentiated by formulas already developed. The results are

$$D_x \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1-x^2}} \qquad (0 < x < 1)$$

$$D_x \operatorname{csch}^{-1} x = \frac{-1}{|x|\sqrt{1+x^2}} \qquad (x \neq 0)$$

We will spare you the details of developing integration formulas based on these derivatives, and simply state the results:

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\cosh^{-1}\frac{a}{|x|} + C = \frac{1}{a}\ln\left|\frac{a-\sqrt{a^2-x^2}}{x}\right| + C$$

$$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\sinh^{-1}\frac{a}{|x|} + C = \frac{1}{a}\ln\left|\frac{a-\sqrt{a^2+x^2}}{x}\right| + C$$

Our last example illustrates the usefulness of hyperbolic functions and their inverses in a concrete setting.

■ Example 4

A body of mass m falls from rest, encountering air resistance that is proportional to the square of the speed. What is the law of motion?

Solution

It is convenient to direct the line of motion downward (so that the velocity is positive). Take the origin at the point where the body starts to fall (and turn on the clock). If s is the position of the body at time t, then $s = 0$ and $v = ds/dt = 0$ at $t = 0$.

The downward force of gravity (mg) and the oppositely directed force of air resistance ($-kv^2$, where $k > 0$) combine to produce a force $F = mg - kv^2$ on the body. Newton's Second Law (force equals mass times acceleration) reduces this to the differential equation

$$m\frac{dv}{dt} = mg - kv^2$$

Separate the variables and integrate:

$$\frac{m\,dv}{mg-kv^2} = dt$$

$$m\int \frac{dv}{mg-kv^2} = \int dt = t + C_1 \qquad (1)$$

The force $F = mg - kv^2$ is always positive (directed downward), so we have

$$\begin{aligned} kv^2 &< mg \\ v^2 &< a^2 \qquad (\text{where } a = \sqrt{mg/k}) \\ |v| &< a \end{aligned}$$

Hence Equation (1) becomes

$$\frac{m}{k}\int \frac{dv}{a^2 - v^2} = \frac{m}{k}\cdot\frac{1}{a}\tanh^{-1}\frac{v}{a} = t + C_1$$

Since $v = 0$ when $t = 0$ we find $C_1 = 0$ and thus

$$\frac{m}{ka}\tanh^{-1}\frac{v}{a} = t$$

$$\tanh^{-1}\frac{v}{a} = bt \qquad (\text{where } b = ka/m)$$

$$\frac{v}{a} = \tanh bt$$

$$v = a\tanh bt$$

Since $v = ds/dt$, we integrate again:

$$\begin{aligned} s &= \int a\tanh bt\, dt = a\int \frac{\sinh bt\, dt}{\cosh bt} \\ &= \frac{a}{b}\int\frac{du}{u} \qquad (u = \cosh bt,\ du = b\sinh bt\, dt) \\ &= \frac{a}{b}\ln|u| + C_2 = \frac{a}{b}\ln\cosh bt + C_2 \qquad (\cosh bt \text{ is positive}) \end{aligned}$$

Since $s = 0$ when $t = 0$, we find $C_2 = 0$ and hence the desired law of motion is

$$s = \frac{a}{b}\ln\cosh bt = \frac{m}{k}\ln\left(\cosh\sqrt{kg/m}\ t\right)$$

Recall from Section 7.7 that Galileo's law of motion (for a body falling from rest in a vacuum) is simply $s = \frac{1}{2}gt^2$. The hypothesis that air resistance is proportional to the square of the speed is not unreasonable, but observe how it complicates matters! The heavy artillery involved in the solution of Example 4 is impressive:

- Newton's Second Law
- Separation of variables in a differential equation
- Integration involving the inverse hyperbolic tangent and the natural logarithm
- The hyperbolic sine, cosine, and tangent (which in turn involve the natural exponential function) ■

Problem Set 9.4

Find each of the following in exact form (no approximations).

1. $\sinh^{-1} 0$
2. $\cosh^{-1} 1$
3. $\tanh^{-1} 0$
4. $\sinh^{-1}(-1)$
5. $\tanh^{-1} \frac{1}{2}$
6. $\cosh^{-1} 0$
7. $\tanh^{-1} 2$
8. If $\sinh x = 2$, find x in terms of logarithms.
9. If $\cosh 2x = 3$, find x in terms of logarithms. Why are there two answers?
10. If $\tanh \sqrt{x} = \frac{3}{5}$, find x in terms of logarithms.
11. Show that the curves $y = \tanh x$ and $y = \operatorname{sech} x$ intersect at (x,y), where $x = \sinh^{-1} 1$ and $y = \tanh(\sinh^{-1} 1) = 1/\sqrt{2}$.
12. If $f(x) = \frac{1}{2}\sinh(x-1)$, find a formula for $f^{-1}(x)$.
13. If $f(x) = 2\tanh(x/2)$, find a formula for $f^{-1}(x)$. What is the domain of f^{-1}?
14. Reflect the graph of $y = \sinh x$ (Section 9.3) in the line $y = x$ to obtain the graph of $y = \sinh^{-1} x$.
15. Reflect the graph of $y = \cosh x$ (Section 9.3) in the line $y = x$ to obtain the graph of the inverse relation. What part of this graph is the graph of $y = \cosh^{-1} x$?
16. Reflect the graph of $y = \coth x$ (Section 9.3) in the line $y = x$ to obtain the graph of $y = \coth^{-1} x$.

Find the derivative of each of the following functions.

17. $f(x) = 3\sinh^{-1} 2x$
18. $f(x) = \cosh^{-1} \sqrt{x}$
19. $f(x) = \tanh^{-1} \dfrac{x}{2}$
20. $f(x) = \coth^{-1}(1/x)$
21. $f(x) = x\sinh^{-1} x - \sqrt{x^2+1}$
22. $f(x) = x\tanh^{-1} x + \dfrac{1}{2}\ln(1-x^2)$
23. $f(x) = \sinh^{-1}(\tan x)$
24. $f(x) = \tanh^{-1}(\sin x)$

Evaluate each of the following integrals.

25. $\displaystyle\int_0^2 \frac{dx}{9-x^2}$
26. $\displaystyle\int_4^6 \frac{dx}{9-x^2}$
27. $\displaystyle\int_0^3 \frac{dx}{x^2-25}$
28. $\displaystyle\int_1^2 \frac{dx}{4x^2-1}$
29. $\displaystyle\int_0^4 \frac{dx}{\sqrt{x^2+4}}$
30. $\displaystyle\int_0^1 \frac{dx}{\sqrt{9x^2+25}}$
31. $\displaystyle\int_2^7 \frac{dx}{\sqrt{x^2-1}}$
32. $\displaystyle\int_2^4 \frac{dx}{\sqrt{9x^2-16}}$
33. $\displaystyle\int_1^2 \frac{dx}{x\sqrt{9-x^2}}$
34. $\displaystyle\int_2^3 \frac{dx}{x\sqrt{16+x^2}}$
35. When a body falls from rest (encountering air resistance proportional to the square of its speed), its velocity at time t is

$$v = a\tanh bt \quad \text{(where } a \text{ and } b \text{ are positive constants)}$$

(See Example 4.)

(a) Explain why v increases with t, but is bounded. Name its least upper bound.

(b) How could this "terminal velocity" have been predicted before the formula for v was found? (Look closely at Example 4!)

36. A skydiver is falling at the rate of 20 ft/sec when her parachute opens; t seconds later her velocity satisfies the equation

$$\frac{dv}{dt} = 32 - \frac{v^2}{50}$$

(a) Solve this initial value problem to show that

$$t = \frac{5}{4}\left(\tanh^{-1}\frac{v}{40} - \tanh^{-1}\frac{1}{2}\right)$$

(b) What terminal velocity does the skydiver approach as time goes on?

(c) How long does it take her to reach 99% of her terminal velocity? (Use a calculator.)

37. Find the derivative of $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ and compare with Example 2.
38. Derive the formula

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + C \qquad (a > 0)$$

by making a substitution that reduces it to the known formula

$$\int \frac{dx}{\sqrt{x^2+1}} = \sinh^{-1} x + C$$

39. Derive the formula $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ as follows.

(a) Write the equation $y = \cosh^{-1} x$ in equivalent hyperbolic form and express the result as an equation that is quadratic in e^y.

(b) Use the quadratic formula to solve for e^y. Include a defense of your choice of the ambiguous sign.

(c) Solve for y.

40. Derive the formula $D_x \cosh^{-1} x = 1/\sqrt{x^2 - 1}$ in two ways, as follows.

(a) Use the formula in Problem 39.

(b) Let $y = \cosh^{-1} x$ and differentiate implicitly in the equivalent hyperbolic equation. Include a defense of your choice of the ambiguous sign.

41. Derive the formula

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C \qquad (x > a)$$

from the known formula

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \cosh^{-1} x + C \qquad (x > 1)$$

42. Show how the formula

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln (x^2 + \sqrt{x^2 - a^2}) + C \qquad (x > a)$$

follows from

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} + C \qquad (x > a)$$

43. Derive the formula

$$\coth^{-1} x = \frac{1}{2} \ln \frac{x + 1}{x - 1}$$

as follows.

(a) Write the equation $y = \coth^{-1} x$ in equivalent hyperbolic form and solve for e^{2y}.

(b) Find y.

44. Derive the formula $D_x \coth^{-1} x = 1/(1 - x^2)$ by using Problem 43.

45. Derive the formulas

$$\int \frac{dx}{a^2 - x^2} = \begin{cases} \dfrac{1}{a} \tanh^{-1} \dfrac{x}{a} + C & (|x| < a) \\ \dfrac{1}{a} \coth^{-1} \dfrac{x}{a} + C & (|x| > a) \end{cases}$$

from the known formulas

$$\int \frac{dx}{1 - x^2} = \begin{cases} \tanh^{-1} x + C & (|x| < 1) \\ \coth^{-1} x + C & (|x| > 1) \end{cases}$$

46. Confirm the formula

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$$

in the case $|x| < a$. (We did the case $|x| > a$ in the text.)

47. Show that

$$\operatorname{sech}^{-1} x = \ln \frac{1 + \sqrt{1 - x^2}}{x} \qquad 0 < x \le 1$$

48. Derive the formula

$$D_x \operatorname{sech}^{-1} x = \frac{-1}{x\sqrt{1 - x^2}}$$

in two ways, as follows.

(a) Use Problem 47.

(b) Use the formula $\operatorname{sech}^{-1} x = \cosh^{-1} (1/x)$.

49. Use the formula $\operatorname{csch}^{-1} x = \sinh^{-1} (1/x)$ to show that

$$D_x \operatorname{csch}^{-1} x = \frac{-1}{|x| \sqrt{1 + x^2}}$$

50. Derive the formula

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a} \cosh^{-1} \frac{a}{|x|} + C$$
$$= \frac{1}{a} \ln \left| \frac{a - \sqrt{a^2 - x^2}}{x} \right| + C$$

51. Derive the formula

$$\int \frac{dx}{x\sqrt{a^2 + x^2}} = -\frac{1}{a} \sinh^{-1} \frac{a}{|x|} + C$$
$$= \frac{1}{a} \ln \left| \frac{a - \sqrt{a^2 + x^2}}{x} \right| + C$$

9.5 The Catenary (Optional)

In Example 4, Section 9.3, we stated that a flexible chain hanging from its ends (and supporting no other weight but its own) takes the shape of the curve $y = a \cosh (x/a)$. This curve is called a **catenary** (from the Latin word for chain). If you are interested in physics, you should enjoy seeing how the equation of a catenary is derived.

Figure 1 shows a chain hanging from two points of support on a horizontal beam. We have chosen the coordinate system so that the lowest point of the chain is at $(0,a)$, where a is a positive constant to be specified later. The point (x,y) is any other point of the chain, the relation between x and y being $y = f(x)$. The problem is to find a formula for $f(x)$.

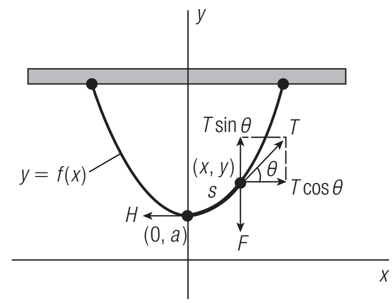

Figure 1 Equilibrium of Forces on a Hanging Chain

We are assuming that the chain is of uniform density δ, that is, it weighs δ lb per ft. If s is the length of the portion from $(0,a)$ to (x,y), the weight of that portion is $F = \delta s$. This downward force is shown as a vector in Figure 1, with initial point at (x,y). The chain hangs motionless because the forces acting on it are in equilibrium. Hence there must be an upward force balancing the downward force F. Imagine two people holding onto the endpoints of the portion shown in Figure 1 (as though the rest of the chain were removed). Each person exerts a tangential pull (called *tension*); these forces, together with the downward force F, keep the chain in the shape shown.

The tension T at (x,y) has horizontal and vertical components $T\cos\theta$ and $T\sin\theta$, respectively, whereas the tension H at $(0,a)$ is entirely horizontal. The upward force balancing F is therefore $T\sin\theta$, while the horizontal forces H and $T\cos\theta$ must cancel each other's effect. In other words, $F = T\sin\theta$ and $H = T\cos\theta$.

A student with no background in physics may not be at home with these ideas. However, we can now forget about physics and concentrate on calculus. The tension T is unknown to us, but we can eliminate it by using the slope m at (x,y):

$$m = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{F/T}{H/T} = \frac{F}{H} = \left(\frac{\delta}{H}\right)s$$

Let $k = \delta/H$ (constant because the density δ and the tension H at the lowest point of the chain are constant). Then $m = ks$. Since $m = dy/dx$, this is a differential equation involving our unknown function $y = f(x)$. It is awkward, however, because s is an integral:

$$s = \int_0^x \sqrt{1 + f'(t)^2}\, dt$$

(See Section 7.3.) We can eliminate the integral by differentiating with respect to x in the equation $m = ks$:

$$\frac{dm}{dx} = k\frac{ds}{dx} = k\sqrt{1 + f'(x)^2}$$

(See the first part of the Fundamental Theorem of Calculus, Section 6.4.) Since $m = f'(x)$, we have $dm/dx = k\sqrt{1 + m^2}$ and things should begin to take shape. For this is a differential equation that can be solved by separating the variables:

$$\frac{dm}{\sqrt{1+m^2}} = k\,dx$$

$$\int \frac{dm}{\sqrt{1+m^2}} = k \int dx$$

You can also see why the catenary is discussed after a section on inverse hyperbolic functions! The integral on the left side fits one of our formulas in Section 9.4; we use it to write

$$\sinh^{-1} m = kx + C_1$$

Since $m = 0$ when $x = 0$ (why?), we find $C_1 = 0$ and hence

$$\sinh^{-1} m = kx$$
$$m = \sinh kx$$

Replacing m by dy/dx, we have another differential equation to solve:

$$\frac{dy}{dx} = \sinh kx$$
$$dy = \sinh kx\,dx$$
$$\int dy = \int \sinh kx\,dx$$
$$y = \frac{1}{k}\cosh kx + C_2$$

Since $y = a$ when $x = 0$ we find $C_2 = a - (1/k)$ and hence

$$y = \frac{1}{k}\cosh kx + \left(a - \frac{1}{k}\right)$$

Our choice of coordinate system in Figure 1 did not specify the value of a. Evidently we can make this number anything we like by moving the x axis up or down. The most convenient choice is the one making $a = 1/k$, in which case our equation for the catenary reduces to $y = a\cosh(x/a)$.

An interesting corollary of the above discussion is that the tension at (x,y) in Figure 1 is $T = \delta y$. (We ask you to show why in the problem set.) Since δy is the weight of y ft of the chain, we could remove one of the upper supports and let y ft hang from a nail driven into the wall at (x,y). (See Figure 2.) The chain would remain in place without slipping over the nail because the tension at (x,y) has been replaced by the weight of the vertical section of length y.

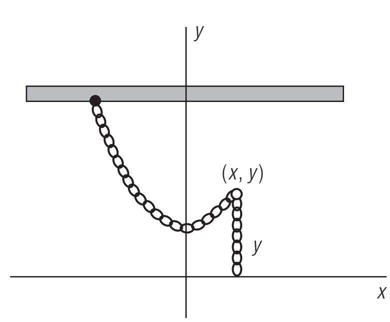

Figure 2 Chain Hanging over a Nail

Problem Set 9.5

1. A rope 50 ft long is strung across a crevasse 40 ft wide, each end being fastened at the same elevation. Find the sag in the rope at its center as follows.

(a) Using the coordinate system in Figure 1, we know that the rope takes the shape of the catenary $y = a \cosh (x/a)$. Explain why the sag at the center is

$$a\left(\cosh \frac{20}{a} - 1\right)$$

(b) Use Problem 53, Section 9.3, to show that a satisfies the equation $\sinh (20/a) = 25/a$.

(c) To solve the equation in part (b), let $t = 20/a$. Confirm that the equation becomes $\sinh t = 1.25t$ and use Newton's Method (Section 5.5) to compute an approximate value of t. *Hint*: A reasonable first guess can be obtained from a sketch of the graphs of $y = \sinh t$ and $y = 1.25t$ on the same coordinate plane. (Why?) Use a calculator to improve it.

(d) The answer in part (c) is $t \approx 1.18$. Use this value to compute the sag.

2. Repeat Problem 1 for a rope 120 ft long strung across a crevasse 100 ft wide.

3. Prove that the tension at (x,y) in Figure 1 is $T = \delta y$, as follows.

(a) Show that $\tan \theta = \sinh (x/a)$.

(b) Why does it follow that $\sec \theta = \cosh (x/a)$?

(c) We know from the text that $H = T \cos \theta$. Explain why this gives

$$T = \delta a \cosh \frac{x}{a} = \delta y$$

4. If the rope in Problem 1 weighs 0.2 lb/ft, what is the tension at its center? at its ends?

5. If the rope in Problem 2 weighs 0.3 lb/ft, what is the tension at its center? at its ends?

6. Suppose that a rope is in the shape of the catenary $y = a \cosh (x/a), -b \le x \le b$, and has length $2s$. Show that the sag in the rope at its center is $\sqrt{a^2 + s^2} - a$.

7. In Problem 1 we have $s = 25$ and $a = 20/t \approx 17$. Use these figures in Problem 6 to compute the sag and compare with the answer in Problem 1(d).

8. In Problem 2 we have $s = 60$ and $a = 50/t \approx 47$. Use these figures in Problem 6 to compute the sag and compare with the answer in Problem 2(d).

True-False Quiz

1. The function $f(x) = \sin x, 0 \le x \le \pi$, has an inverse.
2. $\tan^{-1}(-1) = 3\pi/4$.
3. $\cos (\sin^{-1} x) \ge 0$.
4. $\tan^{-1} x = \sin^{-1} x/\cos^{-1} x$.
5. $\cos (\pi/2 - \sin^{-1} x) = x$.
6. $\tan^{-1} (\tan x) = x$ for all x in the domain of tangent.
7. $\sec^{-1} 5 = \cos^{-1} \frac{1}{5}$.
8. If $f(x) = \sin^{-1} (x/2)$, then $f'(1) = 2/\sqrt{3}$.
9. If $f(x) = \tan^{-1} \sqrt{x}$, then $f'(1) = \frac{1}{2}$.
10. If $f(x) = \sin^{-1} (\cos x)$, then $f'(4) = -1$.
11. If $f(x) = \sec^{-1} \dfrac{1}{x}$, then $f'(x) = \dfrac{-1}{\sqrt{1 - x^2}}$.
12. $D_x(\sin^{-1} x + \cos^{-1} x) = 0$.
13. The inverse of the function

$$F(x) = \int_0^x \frac{dt}{1 + t^2}$$

is $F^{-1}(x) = \tan x, -\pi/2 < x < \pi/2$.

14. $\displaystyle\int_{-2}^{2} \frac{dx}{4 + x^2} = \frac{\pi}{2}$.
15. If $f(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{1 - t^2}}$, then $f^{-1}(\pi/2) = 1$.
16. $\displaystyle\int_{\sqrt{2}}^{2} \frac{dx}{x\sqrt{x^2 - 1}} = \frac{\pi}{12}$.
17. If $dy/dx = 4 + y^2$ and $y = 2$ when $x = 0$, then $y = 2 \tan (x + \pi/4)$.
18. $\displaystyle\int \tan^{-1} x \, dx = x \tan^{-1} x - \tfrac{1}{2} \ln (1 + x^2) + C$.
19. $\sinh (\ln 3) = \frac{4}{3}$.
20. $D_x(\sinh^2 x - \cosh^2 x) = 0$.

21. The range of cosh is $\mathcal{R}$.

22. The domain of $f(x) = \sin^{-1}(\cosh x)$ is $\mathcal{R}$.

23. The hyperbolic tangent is an even function.

24. $0 < \operatorname{sech} x \le 1$ for all x.

25. The function $y = \sinh x$ satisfies the differential equation $y'' + y = 0$.

26. $D_x \ln(\sinh x) = \coth x$.

27. The function $f(x) = 2\sinh x + \cosh x$ has no extreme values.

28. The length of the curve $y = \cosh x$, $-2 \le x \le 2$, is $e^2 - e^{-2}$.

29. $\displaystyle\int \frac{dx}{(e^x + e^{-x})^2} = \frac{1}{4}\tanh x + C.$

30. The hyperbolic cosine has an inverse.

31. $\cosh^{-1} 0 = 1$.

32. $\tanh^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\ln 3$.

33. $D_x \tanh^{-1}(\cos x) = -\csc x$.

34. $D_x \sinh^{-1}(\cot x) = -\csc x$.

35. $\displaystyle\int_2^3 \frac{dx}{1 - x^2} = \tanh^{-1} 3 - \tanh^{-1} 2.$

36. $\displaystyle\int_0^3 \frac{dx}{\sqrt{x^2 + 16}} = \ln 2.$

37. A flexible chain whose ends are attached to two points on a horizontal beam hangs in the shape of the curve $y = a\cosh(x/a)$.

Additional Problems

Find each of the following in exact form. (Do not give approximations.)

1. $\tan^{-1}(-1)$
2. $\cos^{-1}\frac{1}{2}$
3. $\sin^{-1}\left(\sin\dfrac{3\pi}{4}\right)$
4. $\sec^{-1} 1$
5. $\cos(\cos^{-1} 0)$
6. $\sin^{-1}\frac{2}{3} + \cos^{-1}\frac{2}{3}$
7. $\sin^{-1} 2$
8. $\tanh 0$
9. $\cosh 0$
10. $\sinh(\ln 2)$
11. $\tanh^{-1} 0$
12. $\cosh^{-1} 0$
13. When the number 1.5 is entered in a calculator and the inverse sine key is pressed, the display panel flashes (or otherwise indicates nonsense). Why?
14. Most calculators do not have an inverse cotangent key. How would you use one to find $\cot^{-1} 2$?

Find the derivative of each of the following functions.

15. $y = \sin^{-1}(1 - x)$
16. $y = \tan^{-1}|x|$
17. $y = \sec^{-1} 2x$
18. $y = \cos^{-1}(1/x)$
19. $y = \cot^{-1}(x^2 - 1)$
20. $y = \tan^{-1} x + \cot^{-1} x$
21. $y = x\sin^{-1} x$
22. $y = (\tan^{-1} x)^2$
23. $y = x\tan^{-1} x - \ln\sqrt{1 + x^2}$
24. $y = \cosh x^2$
25. $y = \tanh\sqrt{x}$
26. $y = e^{2x}\sinh x$
27. $y = \ln \operatorname{sech} x$
28. $y = \tanh^{-1} x^2$
29. $y = \cosh^{-1} 2x$

30. Explain why the formula

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = -\cos^{-1}\frac{x}{a} + C$$

is correct and use it to evaluate

$$\int_0^1 \frac{dx}{\sqrt{4 - x^2}}$$

Check by means of the standard formula.

Evaluate each of the following.

31. $\displaystyle\int_0^2 \frac{dx}{\sqrt{16 - x^2}}$
32. $\displaystyle\int_0^{\sqrt{3}} \frac{dx}{1 + x^2}$
33. $\displaystyle\int \frac{x\,dx}{\sqrt{1 - x^4}}$
34. $\displaystyle\int_5^6 \frac{dx}{x\sqrt{x^2 - 16}}$
35. $\displaystyle\int \frac{x}{x^2 + 6x}$ *Hint*: Complete the square.
36. $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{9 - \sin^2 x}}$
37. $\displaystyle\int_0^2 \sin\tfrac{1}{2}x\,dx$
38. $\displaystyle\int_0^{\ln 3} \sqrt{\sinh t}\cosh t\,dt$

39. Verify the formula

$$\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x) + C$$

by differentiating the right side.

40. If $dy/dx = 1 + y^2$ and $y = 1$ when $x = 0$, find y as a function of x.

41. If $dy/dx = \sqrt{1 - y^2}$ and $y = 0$ when $x = 1$, find y as a function of x.

42. If $dy/dx = 25 + y^2$ and $y = 5$ when $x = 0$, find y as a function of x.

43. Find the area of the region bounded by the graph of

$$y = \frac{1}{x^2 + 9}$$

and the x axis and the lines $x = \pm 3$.

44. Find the area of the region bounded by the graph of

$$y = \frac{1}{\sqrt{4 - x^2}}$$

and the line $y = 2$.

45. Find the volume of the solid generated by rotating the region under the curve $y = \sqrt{9 - x^2}$, $0 \le x \le 2$, about the x axis.

46. Repeat Problem 45 for rotation about the y axis.

47. If $f(x) = 2 \sinh \sqrt{x}$, find $f^{-1}(x)$.

48. The motion of a spring with displacement x at time t is given by $d^2x/dt^2 = -a^2 x$. If the spring is released from a compressed position of $x = -1$ at $t = 0$, find its law of motion. *Hint*: If $v = dx/dt$, then $dv/dt = v(dv/dx)$.

49. What is the domain of the function $f(x) = \sin^{-1} x + \cos^{-1} x$? By examining $f'(x)$ and drawing appropriate conclusions, write $f(x)$ in simpler form.

50. For what values of x is it true to say that

$$D_x(\sin^{-1} x + \sec^{-1} x) = \frac{1}{\sqrt{1 - x^2}} + \frac{1}{x\sqrt{x^2 - 1}}?$$

51. Show that the function $y = 3 \sinh x - 2 \cosh x$ is always increasing.

52. Find $F'(x)$ if

$$F(x) = x\sqrt{1 - x^2} - 2\int_1^x \sqrt{1 - t^2}\,dt$$

What is a simpler formula for $F(x)$?

53. Show that

$$D_x \cos^{-1}\frac{1}{x} = \frac{1}{|x|\sqrt{x^2 - 1}}$$

and use the result to explain why

$$\sec^{-1} x = \begin{cases} \cos^{-1}\dfrac{1}{x} & \text{if } x \ge 1 \\ 2\pi - \cos^{-1}\dfrac{1}{x} & \text{if } x \le -1 \end{cases}$$

54. Show that

$$\tanh\left(\frac{1}{2}\ln x\right) = \frac{x - 1}{x + 1}$$

55. Prove that the graphs of $y = \sinh x$ and $y = \operatorname{csch} x$ do not intersect at right angles.

56. Find the length of the curve $y = \cosh x$, $0 \le x \le 2$.

57. Find the centroid of the curve $y = \cosh x$, $-1 \le x \le 1$. *Hint*: Use the identity $\cosh^2 t = \frac{1}{2}(\cosh 2t + 1)$ in Problem 16, Section 9.3. Also see Example 4, Section 9.3, where the length of the curve is found to be $2 \sinh 1$.

58. Find the area of the surface generated by rotating the catenary

$$y = a \cosh(x/a) \qquad -a \le x \le a$$

about the x axis. (It can be shown that no other curve with the same endpoints generates a smaller surface area when it is rotated about the x axis.)

59. The region bounded by the curves $y = \tanh x$ and $y = \operatorname{sech} x$ and the y axis is rotated about the x axis. Find the volume of the resulting solid of revolution as follows.

(a) Explain why the volume is given by

$$V = \pi(2 \tanh c - c)$$

where c is the x coordinate of the point of intersection of the curves $y = \tanh x$ and $y = \operatorname{sech} x$.

(b) Show that $c = \ln(1 + \sqrt{2})$.

(c) Conclude that $V = \pi[\sqrt{2} - \ln(1 + \sqrt{2})]$.

60. The graph of a certain function (defined for all x) contains the point (0,1) and at each point (x,y) the square of its slope is $y^2 - 1$. Find a formula for the function.

61. Given $t \in \mathscr{R}$, explain why there is a number x between $-\pi/2$ and $\pi/2$ such that $\sinh t = \tan x$. Then show that

$$\begin{aligned} \cosh t &= \sec x \\ \tanh t &= \sin x \\ \coth t &= \csc x \qquad (t \ne 0) \\ \operatorname{sech} t &= \cos x \\ \operatorname{csch} t &= \cot x \qquad (t \ne 0) \end{aligned}$$

62. The equation $\sinh t = \tan x$, $-\pi/2 < x < \pi/2$, in Problem 61 defines x as a function of t.

(a) What is the formula for x?

(b) Show that $dx/dt = 1/\sec x$.

(c) Separate the variables and integrate in part (b) to obtain

$$\int \sec x \, dx = \sinh^{-1}(\tan x) + C \qquad -\pi/2 < x < \pi/2$$

(d) Use the logarithmic formula for $\sinh^{-1}$ to find

$$\int \sec x \, dx = \ln(\sec x + \tan x) + C \qquad -\pi/2 < x < \pi/2$$

and compare with Example 3, Section 8.5.

63. Show that $\tan^{-1}(\sinh t) = \sin^{-1}(\tanh t)$ for all t in two ways, as follows.

(a) Use Problem 61.

(b) Use differentiation.

64. Show that $\sinh^{-1}(\tan x) = \tanh^{-1}(\sin x)$, $-\pi/2 < x < \pi/2$, in two ways, as follows.

(a) Use Problem 61.

(b) Use differentiation.