KING SAUD UNIVERSITY

Math Department

June 2022

Time: 180mn

Final exam Math106

Question 1(2+3)

a) Let $F(x) = \int_2^{e^x} lnt dt$. Find $F'(1)$.

b) Find the number(s) c that satisfies the conclusion of the mean value theorem for the function $f(x) = \frac{x}{\sqrt{x^2+16}}$ on $[0, 3]$.

Question 2(2+3+3)

a) Evaluate the integral $\int \frac{5^x dx}{5^{2x}+4}$

b) Compute the integral $\int \frac{x-3}{x\sqrt{x^2-25}} dx$

c) Find the indefinite integral $\int \sinh^{-1} x \, dx$

Question 3(3+3+3)

a) Find $\lim_{x\to+\infty}(e^x + x)^{\frac{1}{x}}$

b) Evaluate the integral $\int \frac{\sqrt{9-x^2}}{x^2} dx$

c) Compute $\int \frac{9x^2-20x+10}{x^3-3x^2+2x} dx$

Question 4(3+3+3)

a) Find $\int \frac{dx}{\sqrt{x^2+2x+5}}$

b) Does the integral $\int_1^3 \frac{dx}{\sqrt[3]{x-1}}dx$ converge? Find its value if it does.

c) Sketch the region bounded by the curves $x = y^2$ and $x = 2 - y^2$ and find its area.

Question5(3+3+3)

a) Find the volume of the solid obtained by revolving the region bounded by $y = 4 - x^2$ and $y = 4 - 2x$ about the y-axis.

b) Compute the arc length of the curve $y = coshx$, $0 \leq x \leq 1$.

c) Sketch the region inside $r = 2sin\theta$ and outside $r = 2 - 2sin\theta$ and find its area.

Grading Scheme (Final math106)

Q 1)

a) $F'(x) = xe^{x}$ **(1,5)** so $F'(1) = e$ **(0.5)**

b) $\int_0^3 \frac{xdx}{\sqrt{x^2+16}} = \left[\sqrt{x^2+16}\right]_0^3 = 1$ **(1.5)** so $1=\frac{3c}{\sqrt{c^2+16}}$ and $c = \sqrt{2}$ **(1.5)**

Q2) a) $\int \frac{5^x dx}{5^{2x}+4} = \frac{1}{ln5}\int \frac{du}{u^2+4}$ **(1.5)**

$=\frac{1}{2ln5}\tan^{-1}\left(\frac{5^x}{2}\right) + C$ **(0.5)**

b) $\int \frac{(x-3)dx}{x\sqrt{x^2-25}} = \int \frac{dx}{\sqrt{x^2-25}} - 3\int \frac{dx}{x\sqrt{x^2-25}}$ **(1)**

$= \cosh^{-1}\left(\frac{x}{5}\right) - \frac{3}{5}\sec^{-1}\left(\frac{|x|}{5}\right) + C$ **(2)**

c) $\int \sinh^{-1} x\, dx = x\sinh^{-1} x - \int \frac{xdx}{\sqrt{x^2+1}}$ **(2)**

$= x\sinh^{-1} x - \sqrt{x^2+1} + C$ **(1)**

Q3) a) If $y = (e^x + x)^{\frac{1}{x}}$ then $\lim_{x\to\infty} lny = \lim_{x\to\infty} \frac{ln(e^x+x)}{x}$

$=\lim_{x\to\infty} \frac{e^x+1}{e^x+x}$

$=\lim_{x\to\infty} \frac{e^x}{e^x+1} = 1$ **(2.5)**

Thus $\lim_{x\to\infty}(e^x + x)^{\frac{1}{x}} = e$ **(0.5)**

b) $\int \frac{\sqrt{9-x^2}}{x^2} dx = \int (cot\theta)^2 d\theta \quad x = 3sin\vartheta$ **(1)**

$= \int ((csc\theta)^2 - 1)d\theta = -cot\theta - \theta + C$ **(1)**

$= -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C$ **(1)**

c) $\frac{9x^2-20x+10}{x^3-3x^2+2x} = \frac{5}{x} + \frac{1}{x-1} + \frac{3}{x-2}$ **(2)**

So $\int \frac{9x^2-20x+10}{x^3-3x^2+2x} dx = 5\,ln|x| + ln|x-1| + 3\,ln|x-2| + C$ **(1)**

Q4) a) $\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{du}{\sqrt{u^2+4}}$ $, u = x+1$ **(2)**

$=\sinh^{-1}\left(\frac{x+1}{2}\right) + C$ **(1)**

b) $\int_c^3 \frac{dx}{\sqrt[3]{x-1}} = \frac{3}{2}(2^{\frac{2}{3}} - (c-1)^{\frac{2}{3}})$ **(1.5)**

$\lim_{c\to1^+} \int_c^3 \frac{dx}{\sqrt[3]{x-1}} = 3.\,2^{-1/3}$.

So the integral converges and its value is $3.\,2^{-1/3}$ **(1.5)**

c) Graph **(1)**

We solve $y^2 = 2 - y^2$ so $y = 1 \; or \; y = -1$ **(0.5)**

The area is given by $2\int_{-1}^{1}(1-y^2)dy = \frac{8}{3}$ **(1.5)**

Q5) a) $4 - x^2 = 4 - 2x \Rightarrow x = 0 \; or \; x = 2$ **(0.5)**

$V = \int_0^2 2\pi x(2x - x^2)dx$ **(1.5)**

$=\frac{8\pi}{3}$ **(1)**

b) $L = \int_0^1 \sqrt{1 + (\sinh x)^2}\, dx$ **(1.5)**

$=[sinhx]_0^1 = \frac{e-e^{-1}}{2}$ **(1.5)**

c) Graph **(1)**

From the equation $2sin\theta = 2 - 2sin\theta$ we get $\theta = \frac{\pi}{6}$ $or\ \theta = \frac{5\pi}{6}$ **(0.5)**

$A = \int_{\pi/6}^{5\pi/6}(4sin\theta - 2)\, d\theta = 4(\sqrt{3} - \frac{\pi}{3})$ **(1.5)**