Green's Theorem

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theorem (Green's Theorem)

Let γ be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by γ. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_{\gamma} P(x,y)dx + Q(x,y)dy = \int\int_{D} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

Remark

The notation $\oint_\gamma P(x,y)dx + Q(x,y)dy$ is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve. The Green's Theorem can be written as

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial D} P(x,y)dx + Q(x,y)dy$$

where ∂D is the positively oriented boundary curve of D.

Example

Consider the curve defined by the boudary of the triangle Δ of vertices $(0,0),(1,0),(0,1)$. Use Green's Theorem to calculate a line integral $\int_\gamma x^2 y dx + xy^2 dy$.

$$\begin{aligned} \int_\gamma x^2 y dx + xy^2 dy &= \int_\Delta \left(y^2 - x^2\right) dxdy \\ &= \int_0^1 \left(\int_0^{1-x} (y^2 - x^2) dy \right) dx = 0. \end{aligned}$$

Example

Consider the curve defined by the circle C defined by $x^2 + y^2 = 9$. Use Green's Theorem to calculate a line integral $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$.

$$\begin{aligned} \int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy &= \int_D (7 - 3)dxdy \\ &= 36\pi. \end{aligned}$$

Remark

Another application of Green's Theorem is in computing areas. Since the area of D is $\int\int_D dxdy$, we wish to choose P and Q so that $(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}) = 1$. Hence the area of D id

$$A = \oint_{\partial D} xdy = -\oint_{\partial D} ydx = \frac{1}{2}\oint_{\partial D}(xdy - ydx).$$

For example the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. A paramatrization of the ellipse E is $x(t) = a\cos t$, $y(t) = b\sin t$.

$$A = \frac{1}{2}\oint_E (xdy - ydx) = \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t dt = \pi ab.$$

Exercises

Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

Exercise 1 :
$\int_C (xy^2 dx + 2x^2 y dy)$, where C is the triangle with vertices $(0,0)$, $(2,2)$, and $(2,4)$.

Solution

$$\int_C (xy^2 dx + 2x^2 y dy) = \int_0^2 \int_x^{2x} (2xy) dy dx = \int_0^2 3x^3 dx = 12.$$

Exercise 2 :
$\int_C (\cos y dx + x^2 \sin y dy)$, where C is the rectangle with vertices $(0,0)$, $(5,0)$, and $(5,2)$.

Solution

$$\int_C (\cos y dx + x^2 \sin y dy) = \int_0^5 \int_0^2 (2x+1) \sin y dy dx = 30(1 - \cos 2).$$

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Exercise 3 :
$\displaystyle\int_C (xe^{-2x}dx + (x^4 + 2x^2y^2)dy)$, where C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution

$$\begin{aligned}\int_C (xe^{-2x}dx + (x^4 + 2x^2y^2)dy) &= \int_1^2 \int_0^{2\pi} (4r^3\cos^3\theta + 4r^3\cos\theta\sin^2\theta \\ &= 4\int_1^2 r^4 \int_0^{2\pi} \cos\theta drd\theta = 0.\end{aligned}$$