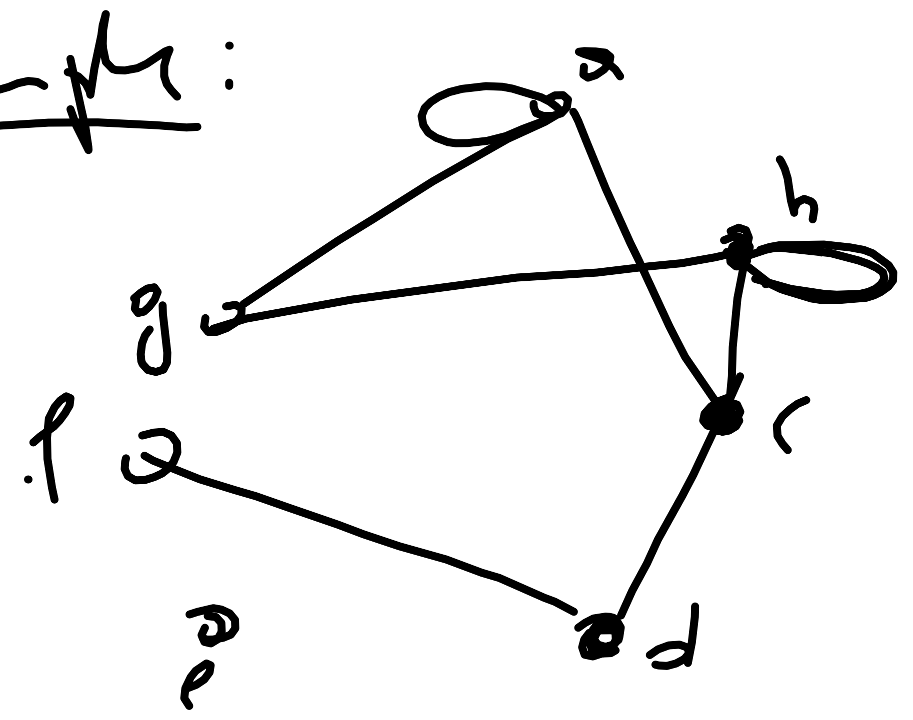
Example:



G

x	a	b	c	d	e	f	g
$d_G(x)$	4	4	3	2	0	1	2

$V = \{a, b, c, d, e, f, g\}$, $|V| = 7$

e : isolated vertex ($d(e) = 0$

f : Leaf or pendant vertex ($d(f) = 1$)

Sequence of degree of G or degree sequence of G

4, 4, 3, 2, 2, 1, 0 → 7 vertices.

$\delta(G) = 0$, $\Delta(G) = 4$

$d(G) = \frac{1}{n} \sum_{x \in V} \deg x = \frac{1}{7}(16) = \frac{16}{7}$

average

we verify: $0 \leq \frac{16}{7} \leq 4$

$\left(\delta(G) \leq d(G) \leq \Delta(G) \right)$

Theorem: Let $G=(V,E)$ be a graph.

1) $\sum_{x\in V} d(x) = 2|E|$.

2) The number of odd vertices in V is even.

Example: $\sum_{x\in V} \deg(x) = 4+4+3+2+2+1+0$
$= 16$
and we have $2|E| = 2(8) = 16$

2) Proof: Let $A = \{x : x\in V \text{ and } \deg(x) \text{ is even}\}$
$B = \{x : x \in V \text{ and } \deg(x) \text{ is odd}\}$
we have $V = A\cup B$ and $A\cap B = \phi$

$$\sum_{x\in V} \deg(x) = \sum_{x\in A} \deg x + \sum_{x\in B} \deg x \overset{1)}{=} 2|E|$$

Then $\sum_{x\in B} \deg x = 2|E| - \sum_{x\in A} \deg x$

$\forall x: (x\in A \longrightarrow \deg x \text{ is even})$

Then $\sum_{x\in A} \deg x$ is even and $2\underbrace{|E|}_{\in \mathbb{N}\cup\{0\}}$ is even

hence $2|E| - \sum_{x\in A} \deg x$ is even

Therefore $\sum_{x\in B} \deg x$ is even

Then $|B|$ is even

Then the number of odd vertices is even.

Proposition: Let $G=(V,E)$ be a graph.

$$\delta(G) \leq d(G) \leq \Delta(G)$$

$\forall x \in V$; $\delta(G) \leq \deg(x) \leq \Delta(G)$

Then $\sum_{x\in V} \delta(G) \leq \sum_{x \in V} \deg x \leq \sum_{x\in V} \Delta(G)$

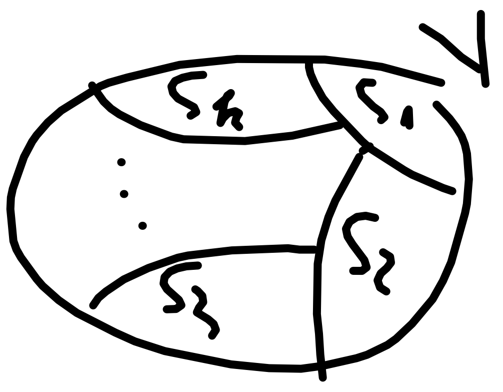
hence $n\,\delta(G) \leq \sum_{x\in V} \deg x \leq n\,\Delta(G)$.

Therefore $\delta(G) \leq \frac{1}{n} \sum_{x\in V} \deg x = d(G) \leq \Delta(G)$

Pigeonhole lemma

$S_1, \ldots, S_k$ is a partition of V, $|V| = n$

(i) $S_i \neq \phi, \forall i \in \{1, \ldots, k\}$; ii) $S_i \cap S_j = \phi, \forall i \neq j$, iii) $V = \bigcup_{i=1}^{k} S_i$)



Proposition: Let $G = (V, E)$ be a graph.

If G is a simple graph, then there are $x \neq y \in V$ such that $\deg x = \deg y$.

Proof: $V = \{x_1, x_2, \dots, x_n\}$; $|V| = n \geq 2$

By contradiction we suppose that, G is a simple graph and $\forall x \neq y \in V$; $\deg x \neq \deg y$.

Suppose that $\deg x_1 < \deg x_2 < \dots < \deg x_n$

$$\sum_{x \in V} \deg x = \deg x_1 + \deg x_2 + \dots + \deg x_n = 2|E|$$

We have in G; $\forall x \in V$; $0 \leq \deg x \leq n-1$

If $\deg x_1 = 0$, then $\deg x_n \leq n-2$

If $\deg x_n = n-1$, then $\deg x_1 \geq 1$

$\deg x \in \{0, 1, \dots, n-2\}$ or $\deg x \in \{1, \dots, n-1\}$.

where $|\{0, \dots, n-2\}| \leq n-1$ and $|\{1, \dots, n-1\}| \leq n-1$

$\exists x \neq y$; $\deg x = \deg y$ $\qquad$ $\exists x \neq y$; $\deg x = \deg y$.

Exercises page 5:

1.

(a) $S = (2,3,3,4,4,5)$ sequence

Suppose that there is a graph $G = (V, E)$, such that

$S = (2,3,3,4,4,5)$ is a sequence degree of G.

First methode

The number of odd vertices (with $\deg x = 3$ or $\deg x = 5$)

↳ 3 (not even) contradiction

(with the number of odd vertices in graph is even)

Then there is no a graph G with S is the sequence degree of G

Second method:

$\sum_{x \in V} \deg x = 2+3+3+4+4+5 = 21 \neq 2|E|$ because $|E| \in \mathbb{N}$.

(b) Suppose that there is a graph $G = (V, E)$

With $S = (2, 3, 4, 4, 4, 6, 6, 6, 9)$ as sequence degree of G

We have $\forall x \in V$; $0 \leq \deg x \leq n-1$, $n = |V|$.

in this case $n = |V| = 9$ and we have x_0; where

$\deg x_0 = 9 > n-1 = 9-1 = 8$

is a contradiction,

Then there is no graph G, with S is the degree sequence of G.

(C) $S = (1,3,3,3)$;

Suppose that there is a graph $G = (V,E)$ such that S is a degree sequence of G.

Let $a \in V$; where $\deg(a) = 1$, where $|V| = 4$

Suppose $V = \{a, x_1, x_2, x_3\}$; $\deg x_1 = \deg x_2 = \deg x_3 = 3$

Then $N(x_1) = \{a, x_2, x_3\}$; $N(x_2) = \{a, x_1, x_3\}$, $N(x_3) = \{a, x_1, x_2\}$

That $\deg a \geq 3 > 1$ contradiction.

(d) Same case of (b)

(e) Same case of (a)

(f) $S = (2, 3, 4, 5, 5, 5)$

Suppose that there is a graph $G = (V, E)$ with S is sequence degree of G.

let $x_1, x_2, x_3 \in V$; $\deg x_1 = \deg x_2 = \deg x_3 = 5$

$|V| = 6$

Then $N(x_i) = V \setminus \{x_i\}$; $\forall i \in \{1, 2, 3\}$ $(\deg x_i = 5)$

Then $\delta(G) \geq 3$ is a contradiction with 2 is a degree of some vertex.

Then there is no a graph G, with S is a sequence degree of G.

2. Let $V = \{s_1, s_2, \dots, s_n\}$; V group of students

$|V| = n \geq 2$

$x \neq y \in V$; $(xy \in E \iff x$ and y are friends$)$.

$G = (V, E)$ is a graph.

Then there are $x \neq y \in V$; such that $\deg x = \deg y$

Then x and y are the same number of friends.

3. let $V = \{c_1, c_2, \dots, c_{15}\}$

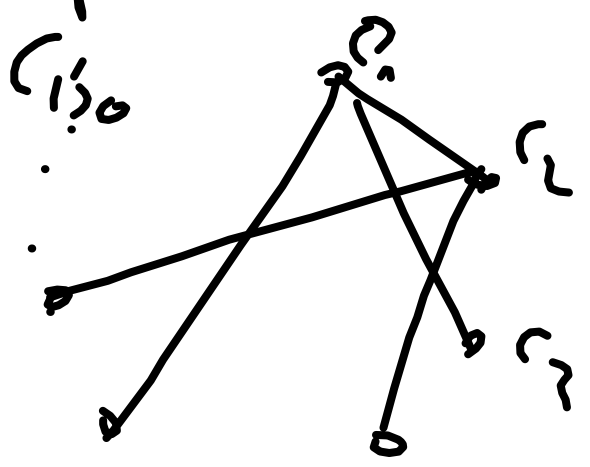
let E be a set when

$x \neq y \in V$; ($xy \in E \iff x$ and y are connected)

Then $G = (V, E)$ is a graph.

Suppose that, is possible each computer connected with exactly 3 others.

Then $\deg x = 3$. ; $\forall x \in V$



we have $\sum_{x \in V} \deg x = 2|E|$

hence $\sum_{x \in V} 3 = 2|E|$, Then $(15) \cdot 3 = 2|E|$

then $|E| = \frac{45}{2} \notin \mathbb{N} \cup \{0\}$ (Contradiction)

Then There is not possible to connected each Computer with exactly 3 others on 15 computers

4. The answer is same with question 3,
by relacing 3 by p and 1's by n; $p < n$.

Exercise 9.1:

1) We have $\sum_{x \in V} \deg x = 2|E|$

Then $48 = 2|E|$, hence $|E| = 24$

2. Let $G = (V, E)$ be a regular graph with type 2 and $|V| = 14$

We have $\sum_{x \in V} \deg x = 2|E|$

Then $\sum_{x \in V} 2 = 2|E|$ because $\deg(x) = 2, \forall x \in V$

Since $|V| = 14$, then $2 \cdot 14 = 2|E|$, therefore $|E| = 14$

3. Let $G = (V, E)$ be a graph with $|E| = 60$ and $|V| = n$.

$\sum_{x \in V} \deg x = 2|E|$, then $\sum_{x \in V} \deg x = 2(60) = 120$.

$\sum_{x \in V} \deg x = 120$

We compare with complete graph K_p

$\left(K_p = (W, F);\ |W| = p \text{ and } \forall x \neq y \in W, xy \in F\right)$

$$|F| = \frac{p(p-1)}{2} \geq |E| = 60$$

$(p^2 - p - 120 = 0)$

Then $p(p-1) \geq 120 \iff p^2 - p - 120 \geq 0$

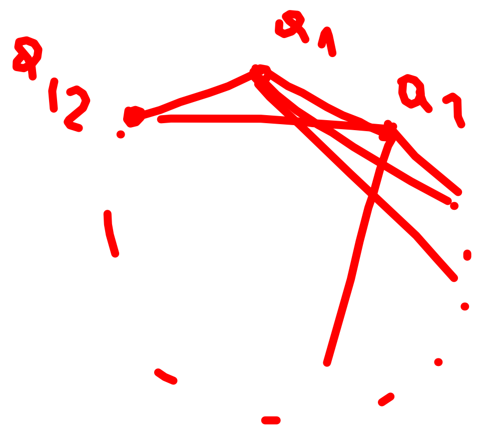
$\Delta = b^2 - 4ac = (-1)^2 - 4(1)(-120) = 1 + 480 = 481 =$

$p_1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{1 - \sqrt{481}}{2} < 0$

$p_2 = \frac{-b + \sqrt{\Delta}}{2} = \frac{1 + \sqrt{481}}{2} \simeq 11.9..$

p		p_1		0		11	p_2	12	
$p^2 - p - 120$	$+$	0	$-$		$-$		0		$+$

$\boxed{p = 12} = n$



K_{12}

$G = K_{12} \setminus \{e_1, e_2, e_3, e_4, e_5, e_6\}$

Subgraph: : Let $G=(V,E)$ and $H=(W,F)$ be two graphs.

1) H is a subgraph of G, if $W \subseteq V$ and $F \subseteq E$.

2) H is a spanning of G, if H is a subgraph of G and $W=V$.

3) Let $X \subseteq V$.

$H=(X, E \cap \{xy : x,y \in X\})$ is a subgraph of G

we said H is a induced subgraph of G on X, and denoted by $H=G[X]$

4) $H=G[V \setminus \{x\}]$; $x \in V$, is a subgraph induced of G on $V \setminus \{x\}$, we denoted by $G-\{x\}$ or $G-x$

In general: Let $X \subseteq V$.

$H=G[V \setminus X]$ is a subgraph induced of G on $V \setminus X$, we denoted $H=G-X$

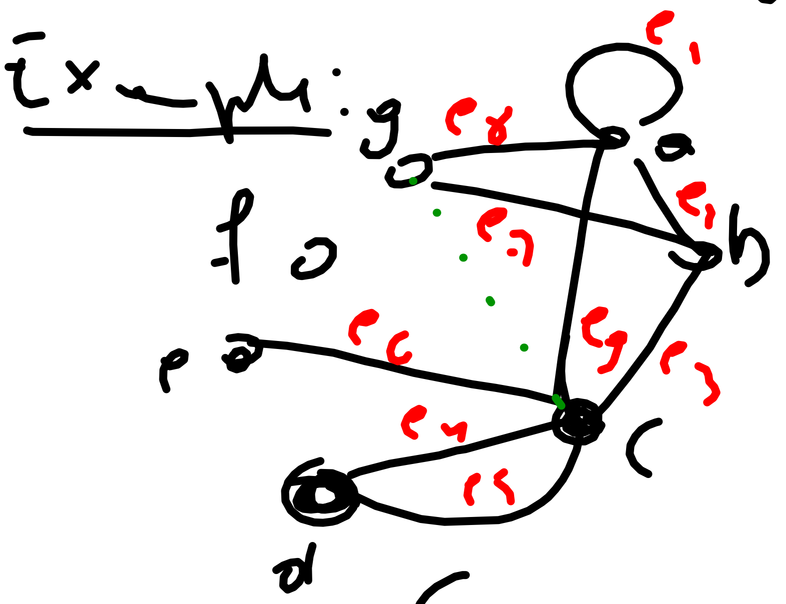
5) Let $E_1 \subseteq E$,

$H=(V, E \setminus E_1)$ is a spanning graph of G.

We denoted $H=G-E_1$

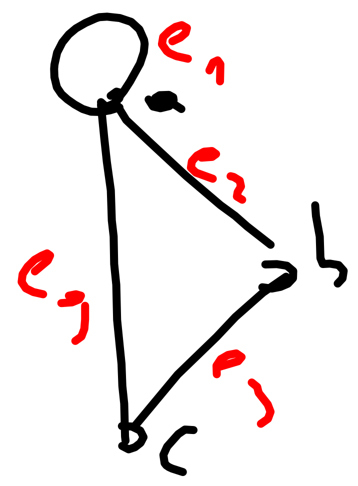
Subgraph: Let $G=(V,E)$ be a graph
Let $H=(W,F)$ be a graph
$H=(W,F)$ is a **subgraph** of G, if $W\subseteq V$ and $F\subseteq E$

Example:

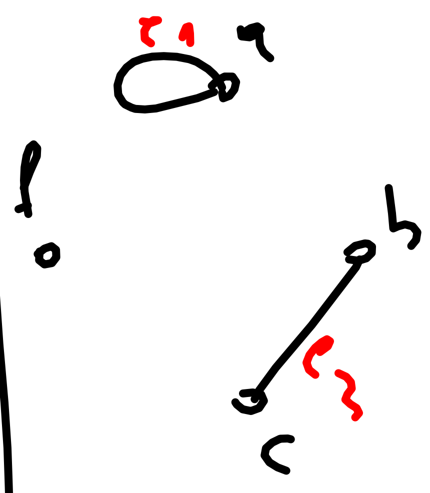


G

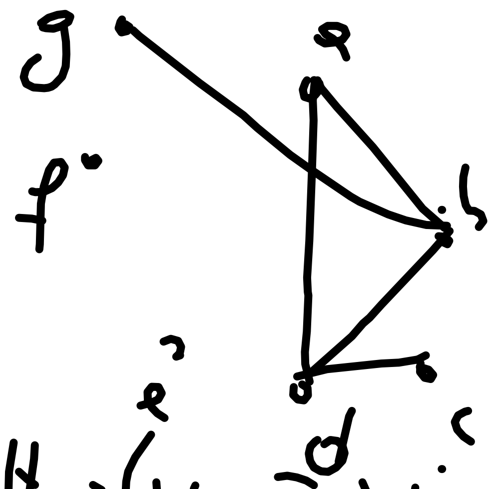
$V=\{a,b,c,d,e,f\}$
$E=\{e_1,e_2,\dots,e_9\}$



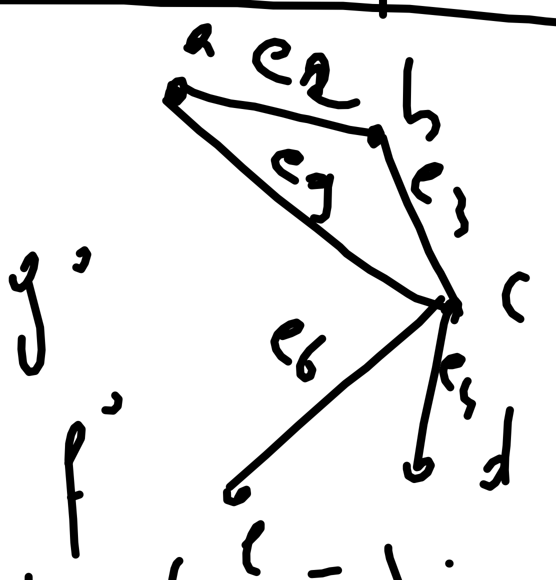
$H_1=(W_1,F_1)$
$W_1=\{a,b,c\}$
$F_1=\{e_1,e_2,e_3,e_9\}$
H_1 is a subgraph of G



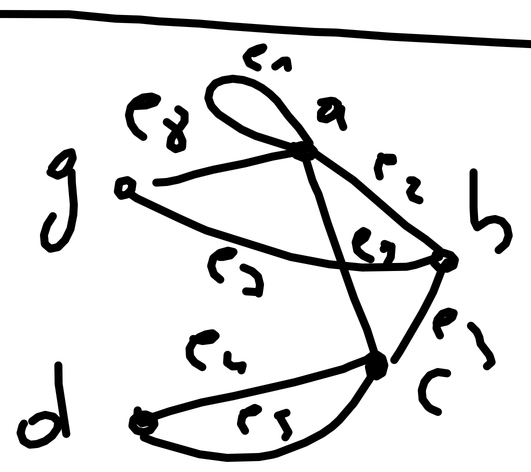
$H_2=(W_2,F_2)$
$W_2=\{a,b,c,f\}$
$F_2=\{e_1,e_3\}$
H_2 is a subgraph of G



$H_3=(W_3,F_3)$ is not a subgraph of G because $ad\notin E$, (then $F_3\not\subseteq E$)



$H_4=(V,F_4)$ is a subgraph of G
H_4: is a spanning of G
$H_4=G-\{e_1,e_5,e_7,e_8\}$

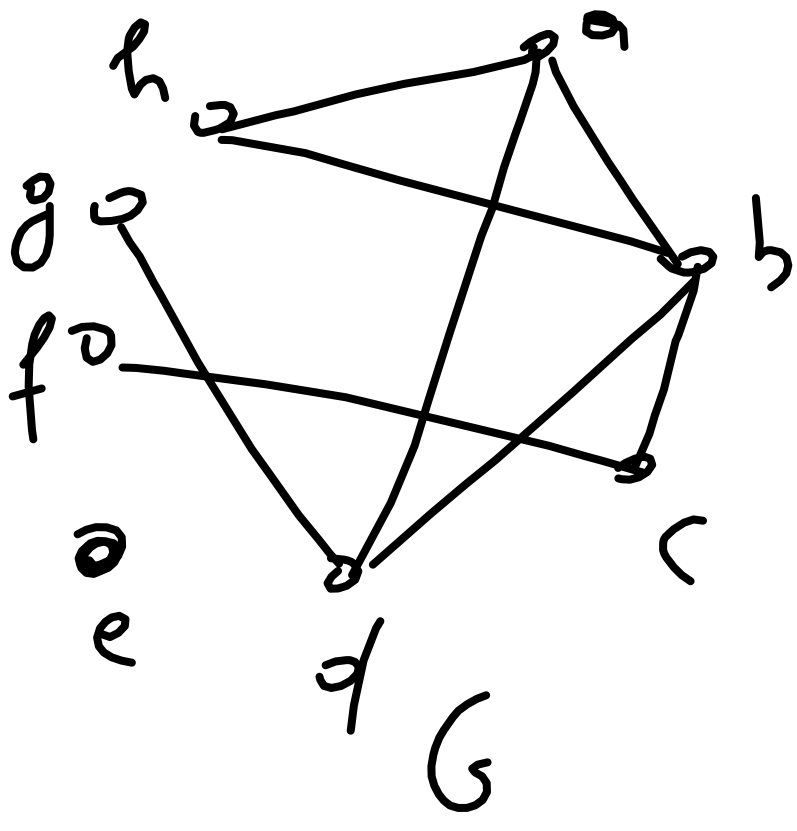


$H_5:=G[\{a,b,c,d,g\}]$ induced subgraph of G on $\{a,b,c,d,g\}$
$H_5=G-\{e,f\}$

Walk - path: Let $G=(V,E)$ be a graph. $u,v \in V$.

- A **walk** W of G from u to v, is a sequence $x_0, x_1, \dots, x_p$ of V where $x_0 = u$ and $x_p = v$ and $x_i x_{i+1} \in E$, $i \in \{0, \dots, p-1\}$
  We denoted $W = (x_0, x_1, \dots, x_p)$
- If $u = v$, A walk W of G from u to v, we said W is a **closed walk**
- A **trail** of G from u to v, if a walk $W = (x_0 = u, x_1, \dots, x_p = v)$ where $\forall i \neq j$, $x_i x_{i+1} \neq x_j x_{j+1}$, $i, j \in \{0, \dots, p-1\}$.
- A **closed trail** or **circuit** $W = (x_0 = u, \dots, x_p = v)$ if W is a trail and $u = v$
- A **path** P of G from u to v, is a walk $P = (x_0 = u, x_1, \dots, x_p = v)$; $x_i \neq x_j$, $\forall i \neq j$
- P is a **closed path** or **cycle** of G on u. if $P = (x_0 = u, x_1, \dots, x_p = u)$ is a path of G from u to u.
- The length of W (walk of G), is the number of edges on W). denoted by $\ell(w) = p$

Example:



Walks of G from c to h.

- $W_1 = (c, b, a, h)$
- $W_2 = (c, f, c, b, d, a, h)$
- $W_3 = (c, b, a, d, b, h)$
- $W_4 = (c, b, a, d, b, h, a, h)$
- $W_5 = (c, b, h)$
- $W_6 = (c, b, a, b, h)$

Trail of G from c to h: $W_1, W_3, W_5,$

Path of G from c to h: W_1, W_5

Remark: let G be a graph, and let W be a walk

1) If W is a path, then W is a trail.

2) In general:

(If W is trail, then W is a path) is false

for example in G: W_3 is a trail, but not a path.

Proposition: Let G be a graph, let $x \neq y \in V(G)$.
If W is a walk from x to y in G, then
there is a path P contains in W, from x to y in G.

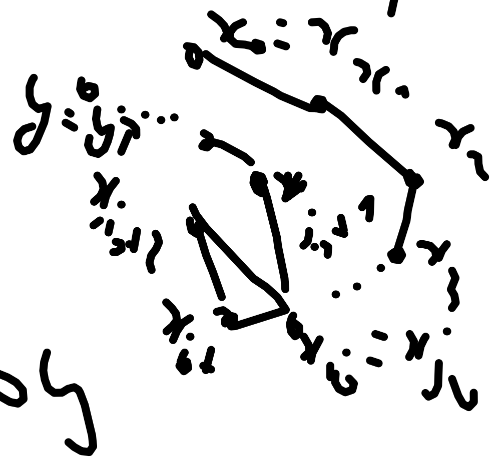
Proof: Let $W = x = x_0, x_1, \dots, x_p = y$ is a walk.

- If $x_i \neq x_j$, $\forall\, i \neq j$, then W is a path ($P = W$)
- If not, $\exists\, i \neq j$ such that $x_i = x_j$, W is not path.

$\longrightarrow$ Suppose that $i_0 \neq j_0$, $x_{i_0} = x_{j_0}$ $(i_0 < j_0)$

where W is:

$W = x_0 = x, x_1, \dots, \underbrace{x_{i_0}, \dots, x_{j_0}}_{=}, \dots, x_p = y.$



Let $P_0 : x_0 = x, x_1, \dots, x_{i_0} = x_{j_0}, x_{j_0+1}, \dots, x_p = y$

P_0 is a walk contains in W, from x to y

by iteration (p is finite), there is a path $P : x = x_0, y_1, \dots, y_\ell = y$

when $x_0, y_1, \dots, y_\ell \in \{x_0, x_1, \dots, x_p\}$

Then P is a path from x to y contains in W in G.

Definition: Let $G = (V, E)$ be a graph.
Let $x, y \in V$

1) x and y are connected, if there is a path from x to y or $x = y$, we denoted $x \sim y$

2) G is connected graph, if $\forall x \neq y$, x and y are connected

If not, G is disconnected graph.

Remark:

1) If x and y are adjacent, then x and y are connected

2) (If x and y are connected, then x and y are adjacents) is false, for example c and h in G (on the example) are connected (w_1 is a walk from c to h) and c and h are not adjacents.

Theorem: Let $G=(V,E)$ be a graph.
The relation $\mathcal{C}$ on V defined by:
$$\forall x,y \in V; (x\,\mathcal{C}\,y) \equiv (x \text{ and } y \text{ are connected}).$$
We have $\mathcal{C}$ is an equivalence relation on V.

Proof: Let $G=(V,E)$ be a graph.

- $x \mathcal{C} x$, because $x=x$, Then $\mathcal{C}$ is reflexive on V
- Let $x,y \in V$, and suppose that $x \mathcal{C} y$, Then there is a path P from x to y in G.
  When $P: x=x_0, x_1, \dots, x_p=y$
  Then $P^{-1}: y=x_p, x_{p-1}, \dots, x_1, x_0=x$ (is the inverse path of P)
  and P^{-1} is a path from y to x, then $y \mathcal{C} x$.

Hence $\mathcal{C}$ is a symmetric relation on V.

- Let $x,y,z \in V$; Suppose that $x \mathcal{C} y$ and $y \mathcal{C} z$

Then there is $P_1: x=x_0, x_1, \dots, x_p=y$ a path from x to y
and there is $P_2: y=y_0, y_1, \dots, y_k=z$ a path from y to z

hence $P = P_1 \cup P_2: x=x_0, x_1, \dots, x_p=y=y_0, y_1, \dots, y_k=z$
is a walk from x to z, by theorem,
there is a path Q from x to z contains in P
Then $x \mathcal{C} z$

Therefore $\mathcal{C}$ is transitive on V

Conclusion: $\mathcal{C}$ is an equivalence relation on V.

Definition: Let $G = (V, E)$ be a graph.

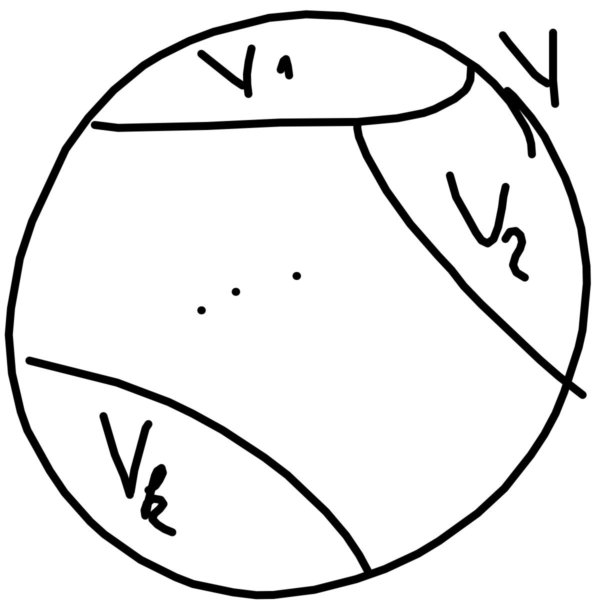
- G is connected $\iff$ $\left(\forall x, y \in V\text{, there is a path from } x \text{ to } y.\right)$
- $\mathcal{C}$: is an equivalence relation on V.
- **In general case:** Let $G = (V, E)$ be a graph.

There is a partition of V (because $\mathcal{C}$ is an equivalence relation)

where $V = V_1 \cup V_2 \cup \dots \cup V_p$, V_i is an equivalence class of $\mathcal{C}$

$\{V_1, V_2, \dots, V_p\}$ is a partition of V

$$\left(\begin{array}{l} \bullet\ \forall i \in \{1, \dots, p\}: V_i \neq \emptyset \\ \bullet\ \forall i \neq j,\ V_i \cap V_j = \emptyset \\ \bullet\ \bigcup_{i=1}^{p} V_i = V \end{array}\right)$$

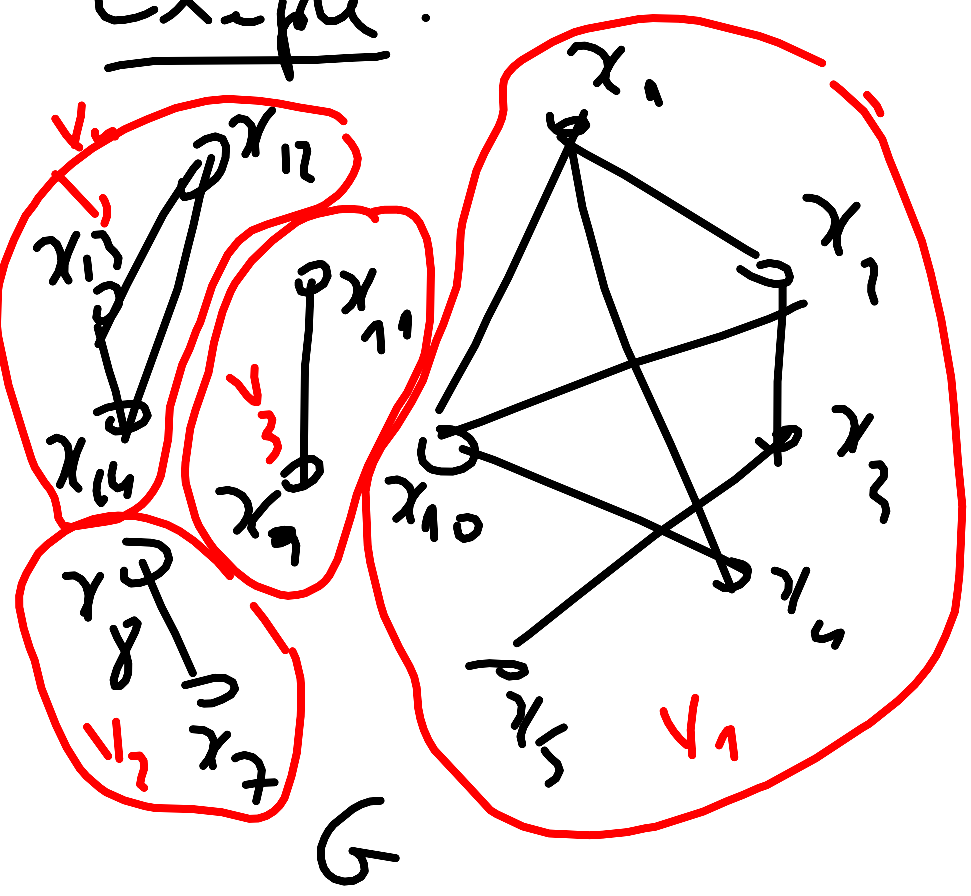


We said each V_i is a connected component of G (and $G[V_i]$, $\forall i \in \{1, \dots, p\}$ is a connected component of G)

Remark: G is connected $\iff$ $\left|\, G \text{ has only one connected component.}\right.$

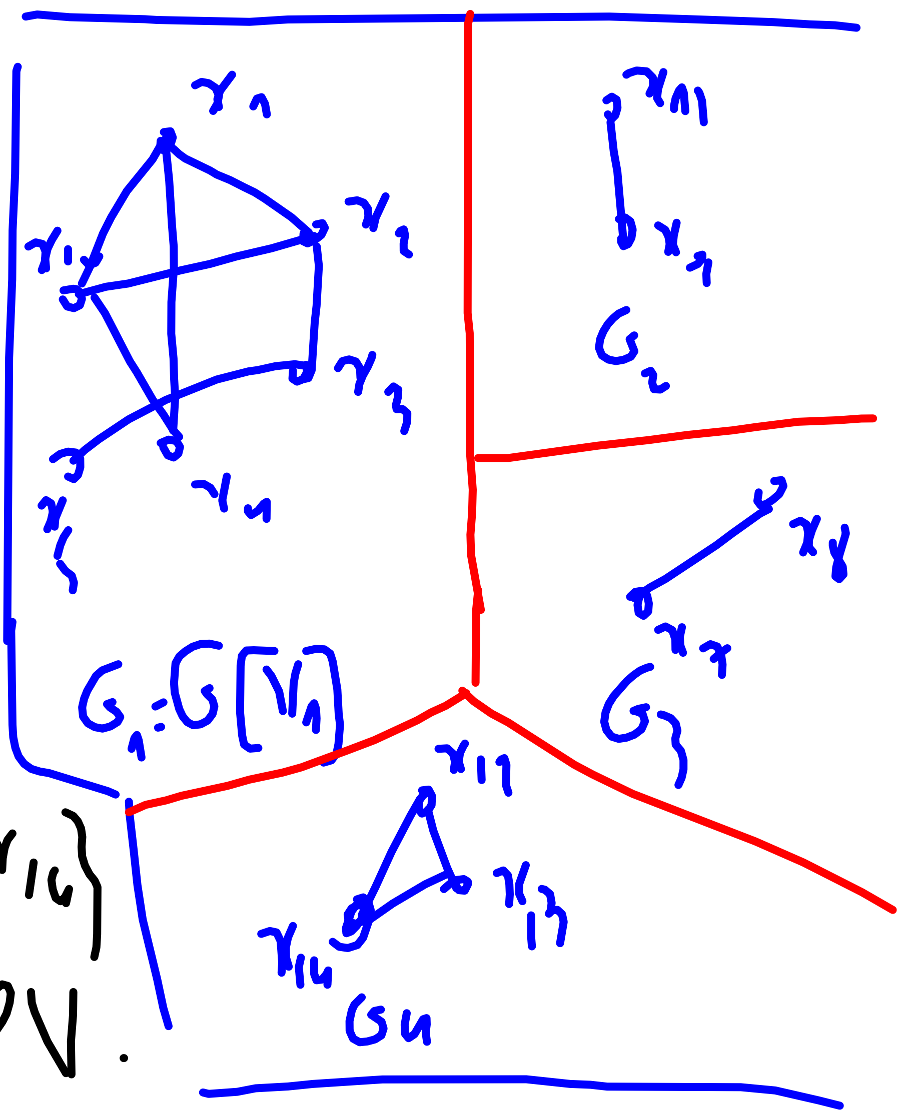
$\iff$ (V is a connected component)

Example:



$V = \{x_1, x_2, \dots, x_{14}\}$.

G



$V_1 = \{x_1, x_2, x_3, x_4, x_5, x_{10}\}$

$V_2 = \{x_7, x_8\}$

$V_3 = \{x_9, x_{11}\}$, $V_4 = \{x_{12}, x_{13}, x_{14}\}$

$\{V_1, V_2, V_3, V_4\}$ is a partition of V.

V_i: is a connected component of G, $\forall i \in \{1,2,3,4\}$

$G[V_i]$: is a connected component of G, $\forall i \in \{1,2,3,4\}$

Particular graphs

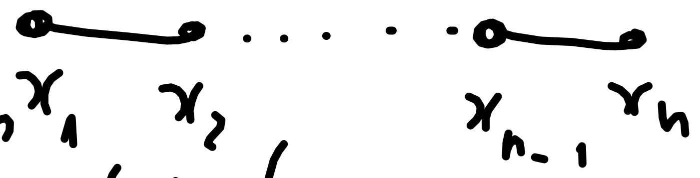
1) Path and cycles.

Let $G=(V,E)$ be a graph

$V=\{v_1,\dots,v_n\}$; $n\geq 1$.

• G is a path if $V=\{x_1,x_2,\dots,x_n\}$ where

$E=\{x_ix_{i+1} : i\in\{1,\dots,n-1\}\}$

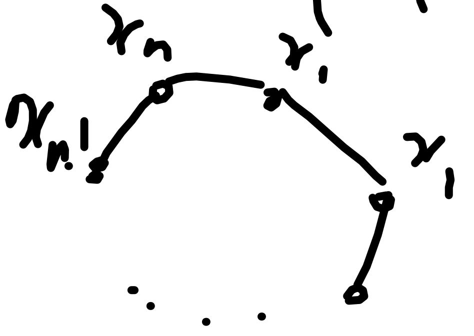


Properties: $|V|=n$

$DEG(G)=(1,1,2,\dots,2)$, $|E|=n-1$

• G is a cycle if $V=\{x_1,\dots,x_n\}$ where $n\geq 3$

$E=\{x_ix_{i+1}, i\in\{1,\dots,n-1\}\}\cup\{x_1x_n\}$, we said G is n-cycle

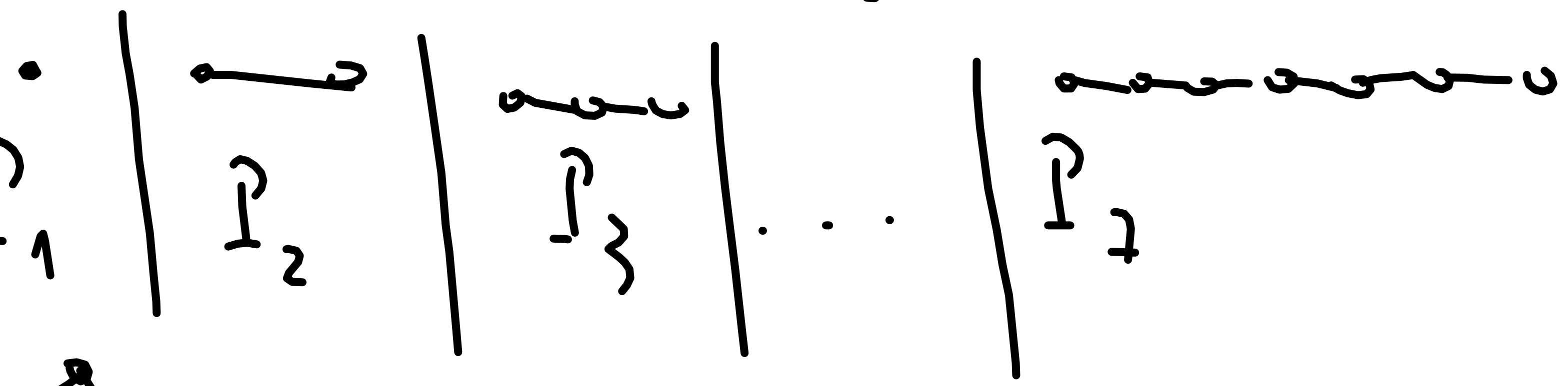


; $|E|=|V|=n$.

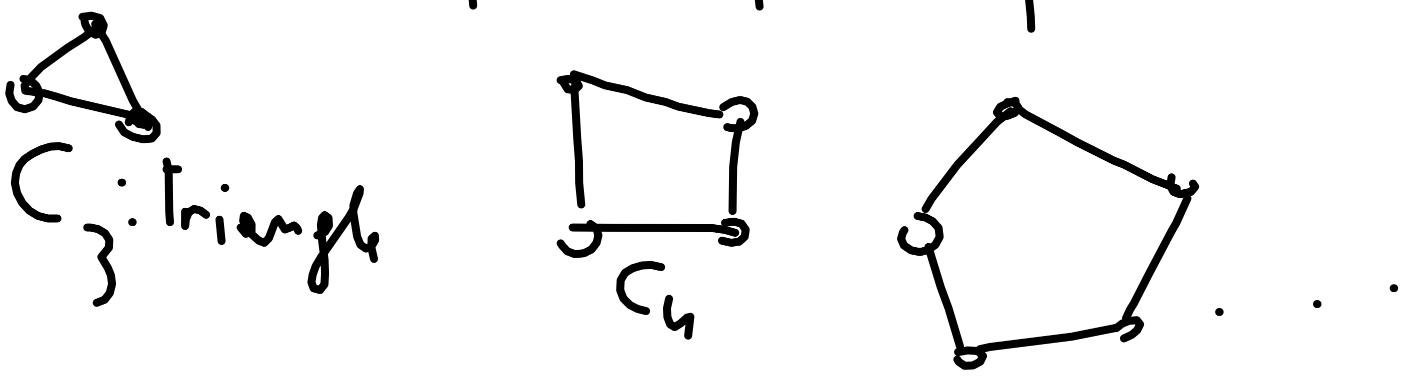
$DEG(G)=(2,2,\dots,2)$.

We denoted P_n: The path with n vertices

C_n: The cycle with n vertices



P_1 | P_2 | P_3 | $\dots$ | P_7



C_3: Triangle

C_4

C_5

$\dots$

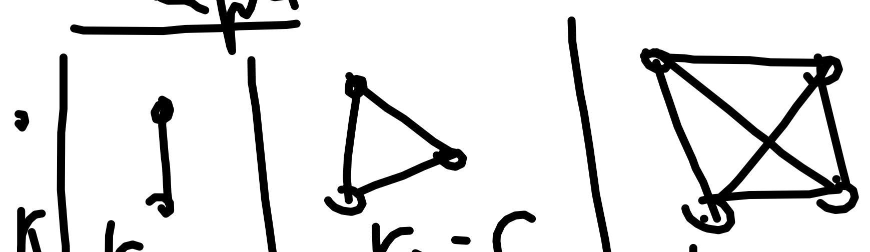
Let $G=(V,E)$ be a graph.

Complete graph:

G is a complete graph if $\forall x \neq y \in V$, $xy \in E$

We denoted $G = K_n$

Example:



K_1 | K_2 | $K_3 = C_3$ | K_4

Null graph:

G is null graph on V with n vertices, if $E = \phi$

We denoted D_n

D_1 | D_2 | D_3 | D_4

$|E| = 0$.

Theorem: Let $G=(V,E)$ be a graph. If G is a complete graph, then

$$|E| = \frac{n(n-1)}{2}, \quad |V| = n.$$

Proof: Let $G=(V,E)$ be a graph. Suppose that G is a complete graph with n vertices, then

$$E = \{xy : x \neq y \in V\}.$$

$$|E| = \binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n\cdot(n-1)}{2}$$

In K_n

$\deg x = n-1$
$\forall x \in V$

Regular graph: Let $G = (V, E)$ be a graph

- G is a regular graph, if there is $r \in \mathbb{N} \cup \{0\}$ such that $\deg(x) = r$

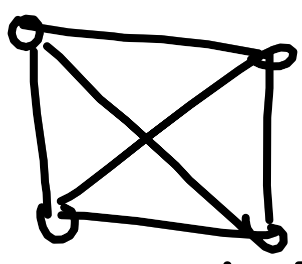
$$DEG(G) = (r, r, \cdots, r)$$

In this case we said G is an r-regular graph.

Example:

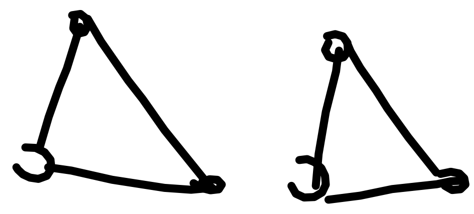
1) The cycle is 2-regular graph

2)



G is 3-regular graph.

3)



G: is 2-regular graph.

4) K_n is a $(n-1)$-regular graph.

Theorem: Let $G = (V, E)$ be a graph, with $|V| = n$. If G is r-regular graph, then $|E| = \frac{n \cdot r}{2}$.

Proof: Assume that G is r-regular graph.

Then $\deg x = r, \forall x \in V$

We have: $\sum_{x \in V} \deg x = 2|E| \Rightarrow \sum_{x \in V} r = 2|E|$

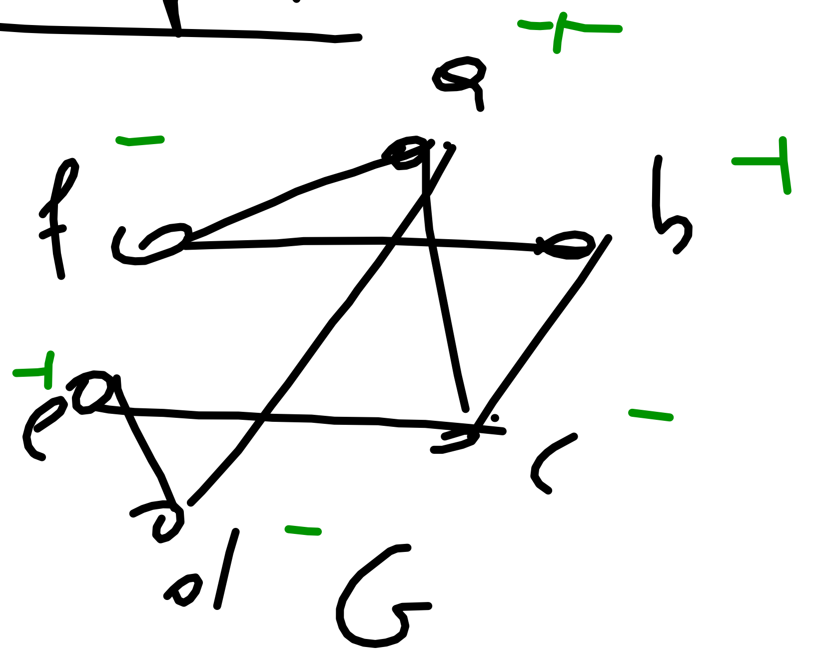
hence $n \cdot r = 2|E| \Leftrightarrow |E| = \frac{n \cdot r}{2}$.

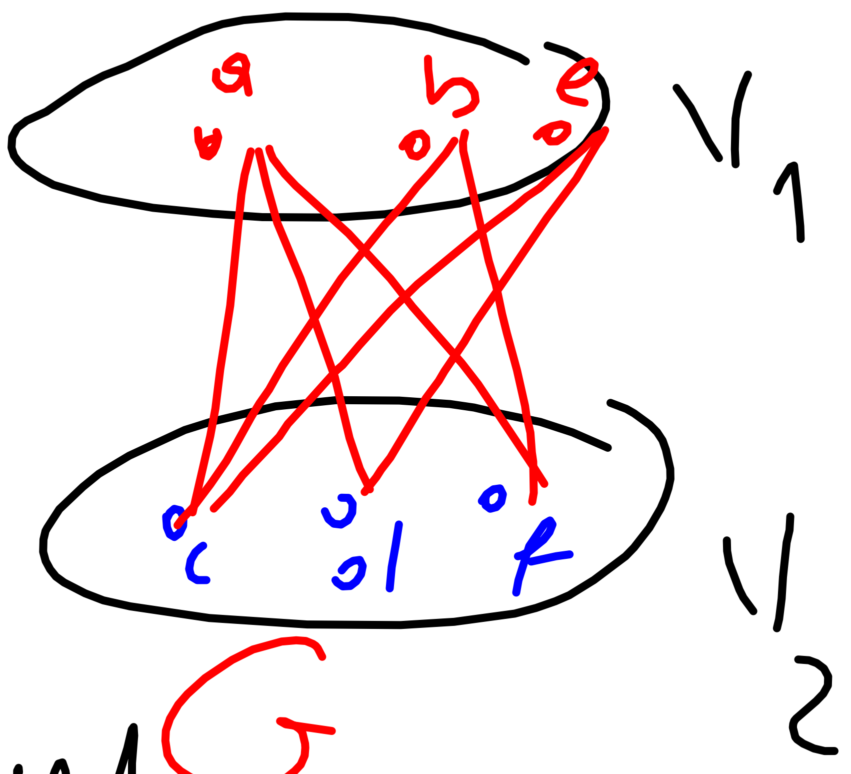
Bipartite graph:

Let $G = (V, E)$ be a graph.
G is bipartite graph, if there is a bipartition $\{V_1, V_2\}$ on V.
$(V = V_1 \cup V_2,\ V_i \neq \emptyset,\ \forall i \in \{1,2\}$, $V_1 \cap V_2 = \emptyset)$.
such that: $\forall x, y \in V_i$:
$xy \notin E$

We denoted: $G = (V_1 \cup V_2, E)$

Example:





G is a bipartite graph.

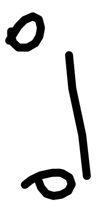
let $G = (V, E)$

G, Complete bipartite graph

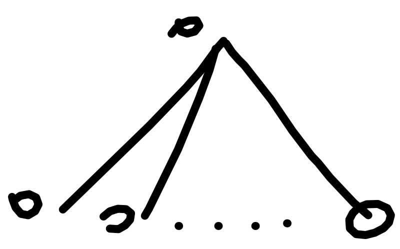
if $G = (V_1 \cup V_2, E)$ is a bipartite graph and $\forall x \in V_1, \forall y \in V_2$ $xy \in E$.

We denoted $G = K_{n,m}$ where $n = |V_1|$ and $m = |V_2|$

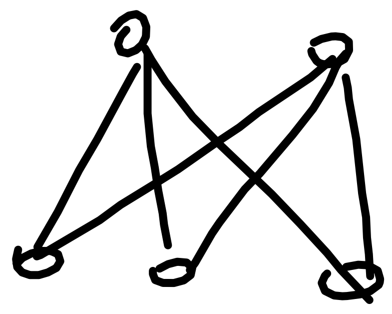
Example:



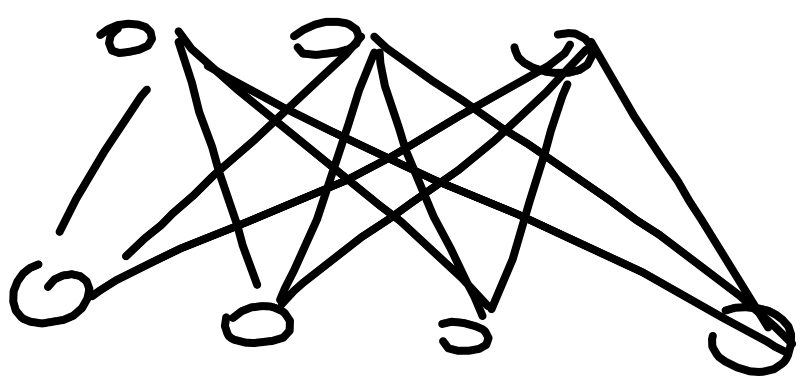
$K_{1,1}$



$K_{1,p}$

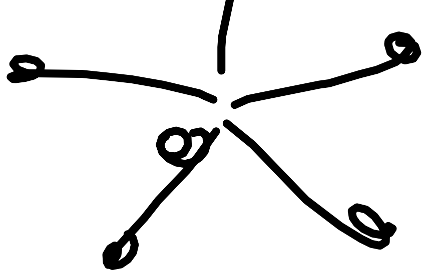


$K_{2,3}$



$K_{3,4}$

Remark: $K_{1,p}$ is a star



Theorem: Let $G = (V, E)$ be a graph.
Let $n, m \in \mathbb{N}$

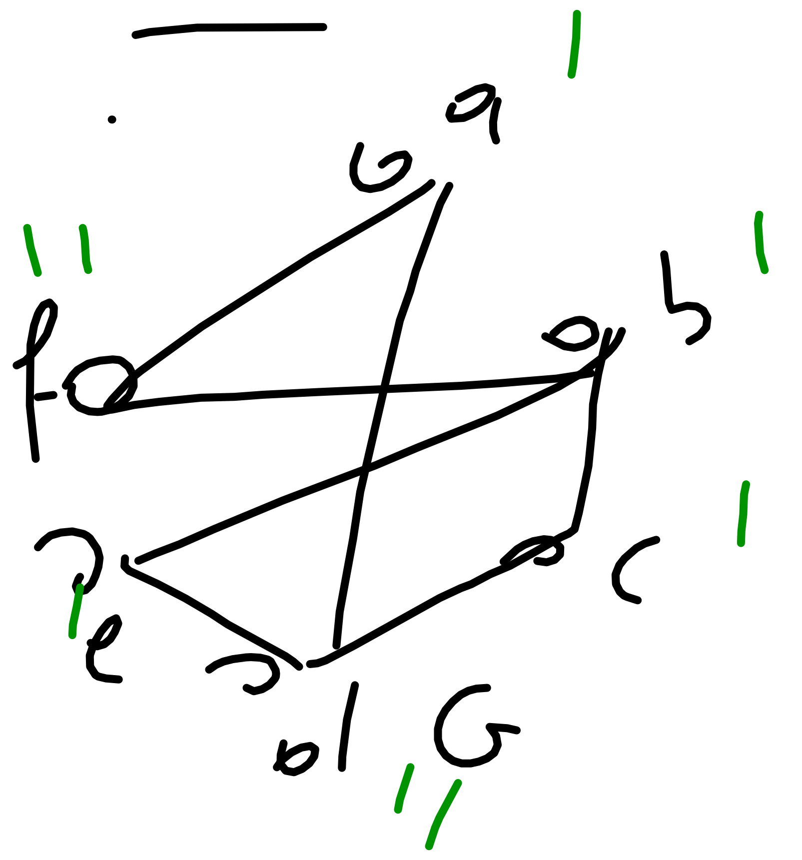
1) If $G = K_{n,m}$, then $|V| = n + m$
$|E| = n \cdot m$

$G = K_{n,m} = (V_1 \cup V_2, E)$; $|V_1| = n$
$\forall x \in V_1$, $\deg x = m$
$\forall x \in V_2$, $\deg x = n$.

2) G is bipartite graph if and only if G has not an odd cycle.

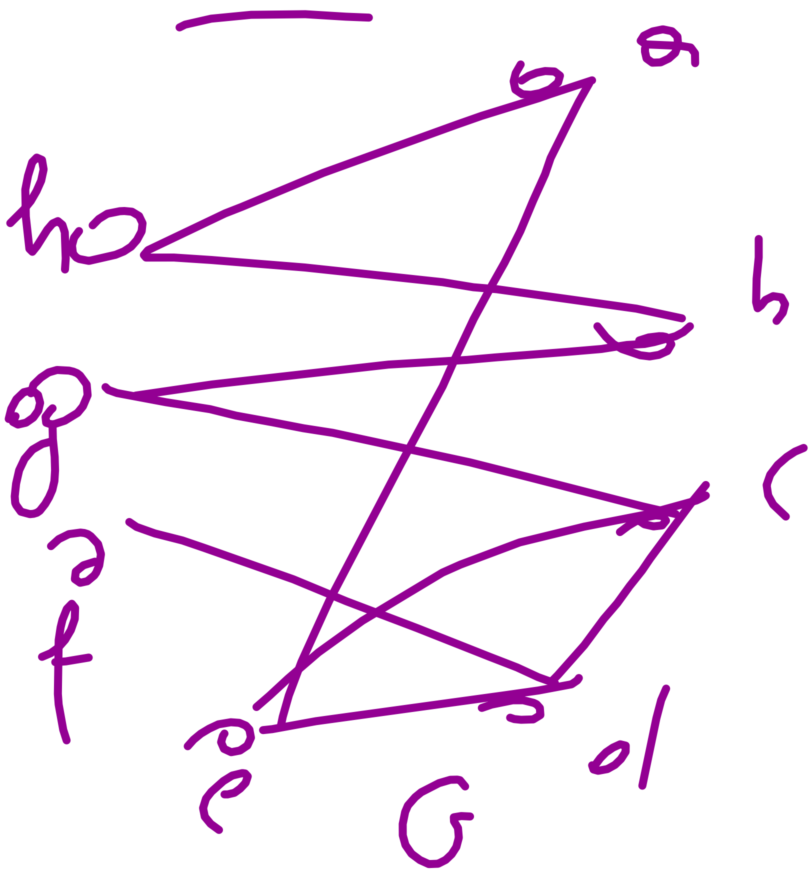
Remark: The odd cycle is not bipartite graph.

Ex:



G is not bipartite graph, because b, e, d, a, f, b is an odd cycle. (5-cycle).

Ex :



G is not bipartite graph, because G has a 3-cycle (e, d, c, e)

Example: Is there a complete bipartite graph $G=(V_1\cup V_2, E)$ such that $|V_1\cup V_2| = 16$ and $|E| = 33$?

Solution: Suppose that there is a complete bipartite graph $G=(V_1\cup V_2, E)$, where $|V_1| = n$ and $|V_2| = m$, we have

$$n+m=16$$
$$n\cdot m=33.$$

- **First method**

$n \in \{1, 3, 11, 33\}$

$\to n=1$, then $m = 16-1 = 15$ and $1\cdot 15 \neq 33$

$\to n=3$, then $m = 16-3 = 13$ and $3\cdot 13 = 39 \neq 33$

$\to n=11$, then $m = 16-11 = 5$, and $5\cdot 11 = 55 \neq 33$

$\to n=33$, then $m = 16-33$: impossible

Conclusion: then there is no a complete bipartite graph such that $|V| = |V_1\cup V_2| = 16$ and $|E| = 33$.

- **Second method:**

$\{n, m\}$ is a solution of the equation

$$X^2 - SX + P = 0 \quad \text{where } \begin{cases} n+m=S \\ n\cdot m=P \end{cases}$$

In this case: $X^2 - 16X + 33 = 0$

$\Delta = b^2 - 4ac = (-16)^2 - 4\cdot 1\cdot 33 =$

- **Third method**

n	1	2	3	4	5	6	7	8
$m = 16-n$	15	14	13	12	11	10	9	8
$n\cdot m$	15	28	39	48	55	60	63	64

Isomorphism:

Def: Let $G = (V, E)$ and $H = (W, F)$ be two graphs.

G and H are isomorphic if there is a function f from V to W $\left(\begin{array}{l} f: V \to W \\ x \mapsto f(x) \end{array}\right)$ a bijection such that

$\forall x, y \in V \ (xy \in E \iff f(x)f(y) \in F)$

In this we said f is an isomorphism from V to W (from G to H) and we denoted $G \simeq H$ or $G \cong H$ or $G \approx H$ or $G \simeq H$

Theorem. Let $G=(V,E)$ and $H=(W,F)$

If G and H are isomorphic

Then 1) $|V|=|W|$ and $|E|=|F|$

2) $DEG(G)=DEG(H)$

3) $\forall\, m \in \mathbb{N}\cup\{0\}$

$$\left|\left\{x : \deg x = m\right\}_{x\in V_G}\right| = \left|\left\{x : \deg x = m\right\}_{x\in W_H}\right|$$

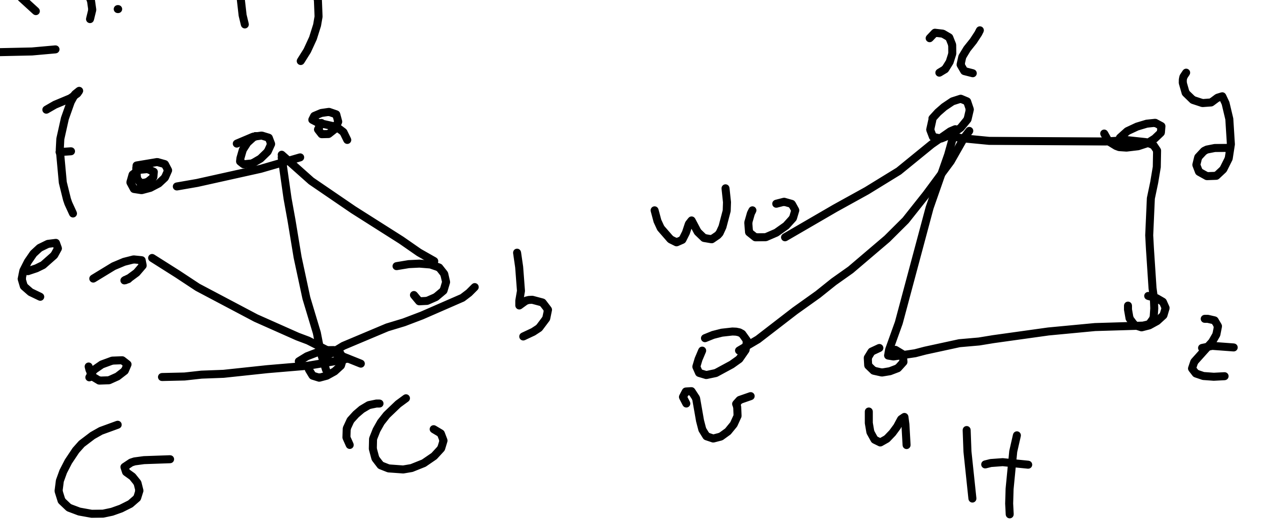
Remark $(p \to q) \equiv (\neg q \to \neg p)$

<u>Remark</u>: 1) If $|E| \neq |F|$ or $|V| \neq |W|$, then $G \ncong H$

2) If $DEG(G) \neq DEG(H)$

then $G \ncong H$

Examples. 1)

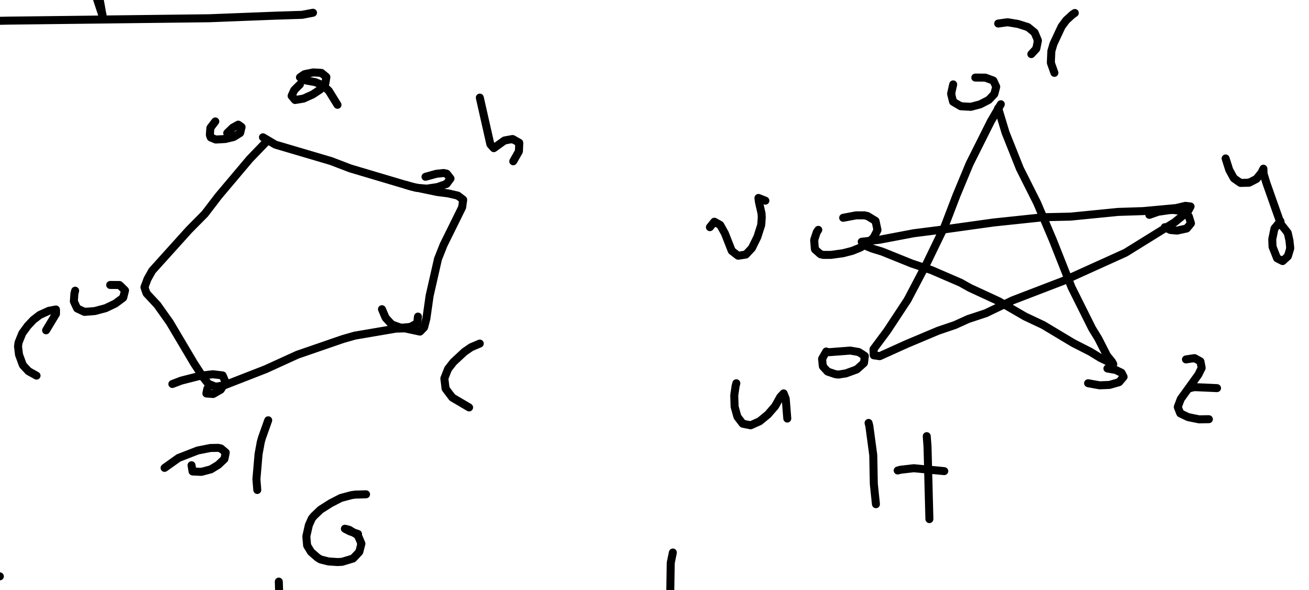


We have $\left| \{ t : \deg_G t = 1 \} \right| = 3$

$\neq \left| \{ t : \deg_H (t) = 1 \} \right| = 2$

Then G and H are not isomorphic.

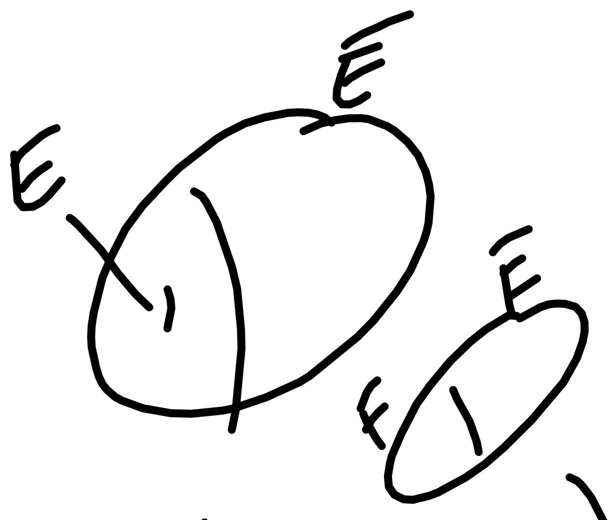
Example 2:



t	a	b	c	d	e
$f(t)$	x	z	v	y	u

we have

$|E(G)| = 5 = |E(H)|$



(the $|E(\bar{G})| = \frac{5(5-1)}{2} - |E(G)| = 10 - 5 = 5$)

$ab \in E(G) \longrightarrow xz \in E(H)$

$bc \in E(G) \longrightarrow zv \in E(H)$

$cd \in E(G) \longrightarrow vy \in E(H)$

$de \in E(G) \longrightarrow yu \in E(H)$

$ed \in E(G) \to yu \in E(H)$

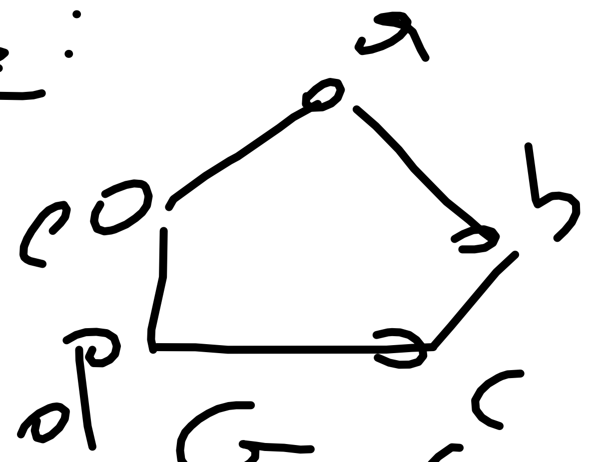
$ux \in E(H)$

Thus G and H are isomorphic

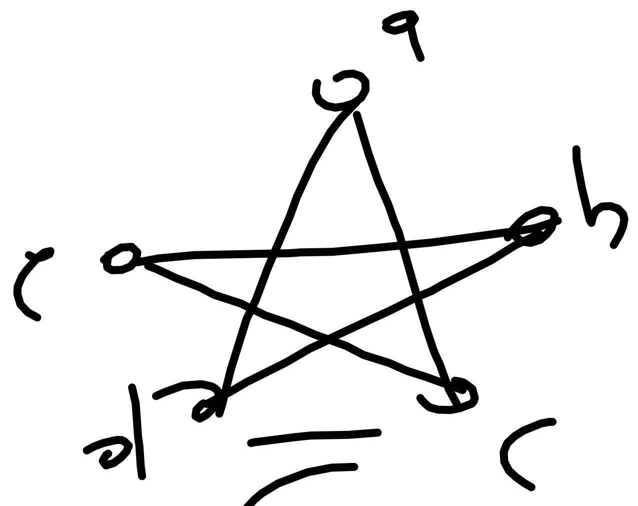
(f is an isomorphism).

Def: Let $G = (V, E)$ be a graph.
G is a self complementary graph; if G and $\bar{G}$ are isomorphic.
$\bar{G} = (V, \bar{E})$: The complement graph of G

Example:



$G = C_5$

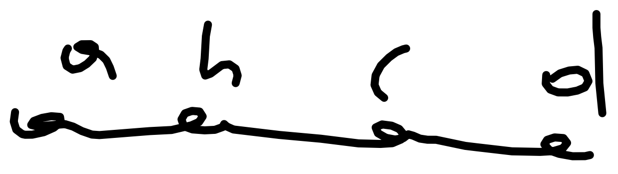


$\bar{G}$

We know G and $\bar{G}$ are isomorphic.
Then G is a self complementary graph

(C_5 is a self complementary graph.)

Example:



P_4

$\bar{P}_4$

$\bar{P}_4$

$\bar{P}_4 \cong P_4$.

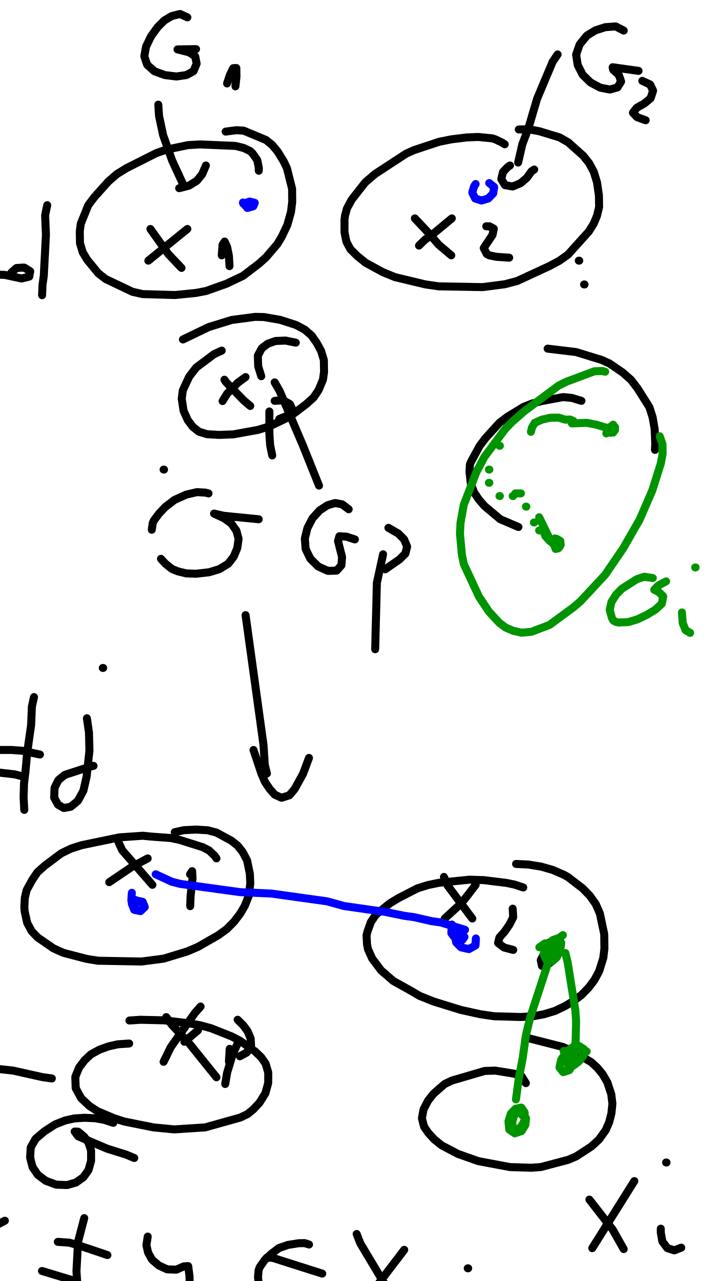
Then P_4 is a self complementary graph.

Theorem: Let $G=(V,E)$ be a graph.
We have G or $\bar{G}$ is connected

(Rk: Is not possible to find a graph G and $\bar{G}$ are both disconnected)

Proof: • If G is connected, then we have a result.

• If G is disconnected then they are p connected components $X_1, \dots, X_p$; $p \geq 2$.



→ $x \in X_i$, $y \in X_j$, $i \neq j$

since $xy \notin E(G)$
then $xy \in E(\bar{G})$

→ Let $i \in \{1, \dots, p\}$; $x \neq y \in X_i$

$\forall j \neq i \in \{1, \dots, p\}$, let $z \in X_j$

$xz \in E(\bar{G})$ and $zy \in E(\bar{G})$

hence x, z, y is a path on $\bar{G}$

hence $\bar{G}$ is connected graph.

Conclusion: G or $\bar{G}$ is connected graph.

Theorem: Let $G=(V,E)$ be a graph with $|V|=n$

If G is a self complementing graph, then

1) G and $\bar{G}$ are connected graph.

2) $|V| = \begin{cases} 4p & ; p \in \mathbb{N} \\ 4p+1, & p \in \mathbb{N} \cup \{0\} \end{cases}$

Proof: 1) We know G or $\bar{G}$ are connected (previous theorem).

Without lost of generality, we suppose that G is connected, since G and $\bar{G}$ are isomorphic (G is self complementary) then $\bar{G}$ is connected.

Therefore G and $\bar{G}$ are connected.

2) Let G is self complementary graph then $|E(G)| = |E(\bar{G})|$ and we have $|E(G)| + |E(\bar{G})| = \frac{n(n-1)}{2}$

Then $2|E(G)| = \frac{n(n-1)}{2}$

hence $|E(G)| = \frac{n(n-1)}{4} \in \mathbb{N} \cup \{0\}$

if $n = 4p$, $p \in \mathbb{N}$, $\frac{n(n-1)}{4} \in \mathbb{N} \cup \{0\}$

$n = 4p+1$, then $n-1 = 4p$, hence $\frac{n(n-1)}{4} \in \mathbb{N}$

$n = 4p+2$, then $\frac{(4p+2)(4p+1)}{4} = \frac{(2p+1)(4p+1)}{2} \notin \mathbb{N}$

$n = 4p+3$, then $\frac{n(n-1)}{2} = \frac{(4p+3)(4p+2)}{4}$

$= \frac{(4p+3)(2p+1)}{2} \notin \mathbb{N} \cup \{0\}$

hence $n = \begin{cases} 0 \\ 1 \end{cases} \pmod 4$

Exercise: Prove that:

1) C_n is a self complementary graph if and only if $n=5$.

2) P_n is a self complementary if and only if $n=1$ or $n=4$.

Proof: 1) "$\Rightarrow$" C_n is self complementary.

$n = 4p$ or $n = 4p+1$, $p \in \mathbb{N}\cup\{0\}$

$$|E(C_n)| = |\overline{E}(C_n)|$$

$$\underbrace{|E(C_n)|}_{n} \qquad \underbrace{|\overline{E}(C_n)|}_{\frac{n(n-1)}{2} - n = \frac{n^2-3n}{2}}$$

$$n^2 - 3n = 2n \Leftrightarrow n^2 - 5n = 0$$
$$\Leftrightarrow n(n-5) = 0$$
$$\Leftrightarrow n = 0 \text{ or } n = 5 \ (n \neq 0)$$
$$\Rightarrow n = 5$$

"$\Leftarrow$" When $n=5$ we have (in example)

$C_5 \simeq \overline{C_5}$, hence C_5 is self complementary

2) "$\Rightarrow$" P_n is self complementary,

Then $|E(P_n)| = |\overline{E}(P_n)|$

$$\underbrace{|E(P_n)|}_{n-1} \qquad \underbrace{|\overline{E}(P_n)|}_{\frac{n(n-1)}{2} - (n-1)}$$

then $n(n-1) = 4(n-1)$

$\Leftrightarrow (n-1)(n-4) = 0$

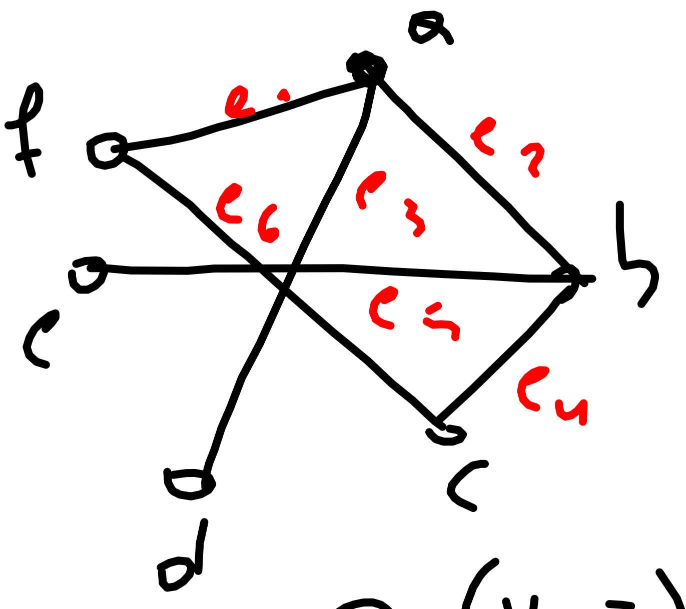
$\Leftrightarrow n = 1$ or $n = 4$

"$\Leftarrow$" P_4 is self complementary (in an example)

$P_1 \quad \bullet \simeq \bullet$ are isomorphic.

Then P_1 (resp. P_4) P_1 is self complementary

Adjacency matrix and length of path



Adjacency matrix of G

$$A(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{matrix} a & b & c & d & e & f \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$G = (V, E)$,

$V = \{a, b, c, d, e, f\}$

Incidency matrix

$$I(G) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Theorem: Let $G=(V,E)$ be a graph.

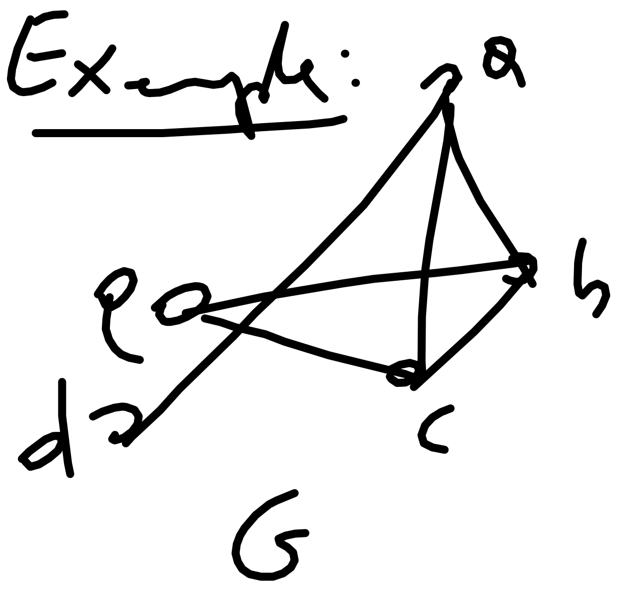
$V=\{v_1,\dots,v_n\}$; $A=(a_{ij})_{1\leq i,j\leq n}$ the adjacency matrix of G; $\left(a_{ij}=\begin{cases}1 & \text{if } v_iv_j\in E\\ 0 & \text{if } v_iv_j\notin E\end{cases};\ i\neq j\right)$

Let $k\in\mathbb{N}$

$A^k=(b_{ij})_{1\leq i,j\leq n}$

b_{ij}: The number of walk from v_i to v_j with lenght k

Example:



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

How many walks from a to d, with lenght 4.

$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 0 & 2 \\ & & & 1 & \\ & & & 1 & \\ & & & 1 & \\ & & & 0 & \end{bmatrix}$$

$$M^4 = M^2 \cdot M^2 = \begin{bmatrix} 3 & 1 & 1 & 0 & 2 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \cdot \begin{bmatrix} \cdot & \cdot & \cdot & 0 & \\ & & & 1 & \\ & & & 1 & \\ & & & 1 & \\ & & & 0 & \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

There is 2 walks from a to d with lenght 4.

a b c a d $\longrightarrow$ not path | trail

a c b a d $\longrightarrow$ not path. | trail

There is 0 paths from a to d with lenght 4.

There is 2 trails from a to d ______