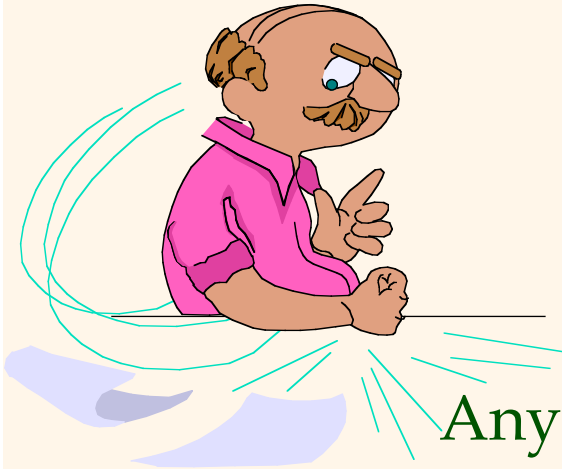# *Gauss-Jordan Elimination*

Any linear system must have exactly one solution, no solution, or an infinite number of solutions. Just as in the 2X2 case, the term **consistent** is used to describe a system with a **unique solution**, **inconsistent** is used to describe a system with no solution, and **dependent** is used for a system with an infinite number of solutions.

Karl Frederick Gauss:

❖ At the age of seven, **Carl Friedrich Gauss** started elementary school, and his potential was noticed almost immediately. His teacher, Büttner, and his assistant, Martin Bartels, were amazed when Gauss summed the integers from 1 to 100 instantly by spotting that the sum was 50 pairs of numbers each pair summing to 101.

Matrix representation of a system of linear equations

$$2x + 3y + 5z = 7$$

$$-3x + y + 4z = 0$$

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ -3 & 1 & 4 \end{pmatrix}, \quad \text{for the coefficients}$$

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \text{for the unknowns}$$

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{B}$$

$$\mathbf{B} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}, \quad \text{for the constant terms}$$

Matrix representation of a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

.

.

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{X} = \mathbf{B}$$

The augmented matrix

❖ For any system of linear equations we can form the so called **augmented matrix**. This is a matrix which contains both the coefficients of the unknown quantities and the constants terms.

❖ With this matrix we can perform all the actions which we perform on the equations when we try to solve a system.

❖ We can do the following actions on this matrix:

❖ A) We can interchange the positions of two rows. Ri ↔ RLj

❖ B) We can multiply one row with a number. Ri → λRi

❖ C) We can add to one row another row multiplied by a number. Ri → Ri+μRj

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right)$$

Matrix representations of consistent, inconsistent and dependent systems

❖ The following matrix representations of three linear equations in three unknowns illustrate the three different cases:

❖ Case I : consistent

$$\begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & 4 \\ 0 & 0 & 1 & | & 5 \end{pmatrix}$$

❖ From this matrix representation, you can determine that

❖ x = 3, y = 4 and z= 5

Matrix representations of consistent, inconsistent and dependent systems

❖ Case 2:

❖ Inconsistent case:

$$\begin{pmatrix} 1 & 2 & 3 & \bigg| & 4 \\ 0 & 0 & 0 & \bigg| & 6 \\ 0 & 0 & 0 & \bigg| & 0 \end{pmatrix}$$

❖ From the second row of the matrix, we find that

❖ 0x + 0y +0z =6   or
        0 = 6

an impossible equation. From this, we conclude that there are no solutions to the linear system.

Matrix representations of consistent, inconsistent and dependent systems

❖ Case 3:

❖ Dependent system

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

❖ When two rows of a matrix representation consist entirely of zeros, we conclude that two the linear equations were identical and therefore, the system is dependent.

❖ (true for case 3 but not in general)

Reduced row echelon form

A matrix is said to be in **reduced row echelon form** or, more simply, in **reduced form**, if :

👉 Each row consisting entirely of zeros is below any row having at least one non-zero element.

👉 The leftmost nonzero element in each row is 1.

👉 All other elements in the column containing the leftmost 1 of a given row are zeros.

4. The leftmost 1 in any row is to the right of the leftmost 1 in the row above.

Examples of reduced row echelon form:

a)
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array} \right)$$

b)
$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

c)
$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{array} \right)$$

Solving a system using Gauss-Jordan Elimination

Problem:

Solve: $x + y - z = -2$
$2x - y + z = 5$
$-x + 2y + 2z = 1$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{bmatrix}$$

1. We begin by writing the system as an augmented matrix

❖ We already have a 1 in the diagonal position of first column.

❖ Now we want 0's below the 1.

❖ The first 0 can be obtained by multiplying row 1 by -2 and adding the results to row 2 :

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ -1 & 2 & 2 & 1 \end{bmatrix}$$

Example continued:

❖ The second 0 can be obtained by adding row 1 to row 3:

Row 1 is unchanged

Row 2 is unchanged

Row 1 is added to Row 3

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & 1 & -1 \end{bmatrix}$$

❖ Moving to the second column, we want a 1 in the diagonal position (where there is now –3). We get this by dividing every element in row 2 by -3:

Row 1 is unchanged

Row 2 is divided by –3

Row 3 is unchanged

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & 1 & -1 \end{bmatrix}$$

Example continued:

❖ To obtain a 0 below the 1 , we multiply row 2 by -3 and add it to the third row:

Row 1 is unchanged
Row 2 is unchanged
-3 times row 2 is added to row 3

❖ To obtain a 1 in the third position of the third row, we divide that row by 4. Rows 1 and 2 do not change.

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

13

Example continued:

We can now work upwards to get zeros in the third column above the 1 in the third row. We will add R3 to R2 and replace R2 with that sum **and** add R3 to R1 and replace R1 with the sum . Row 3 will not be changed. All that remains to obtain reduced row echelon form is to eliminate the 1 in the first row, 2nd position.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Example continued:

To get a zero in the first row and second position, we multiply row 2 by -1 and add the result to row 1 and replace row 1 by that result. Rows 2 and 3 remain unaffected.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Final result:

We can now "read" our solution from this last matrix.

We have x = 1, y = -1 and z = 2. Written as an ordered triple, we have (1, -1, 2). This is a **consistent** system with a unique solution.

Example 2

❖ Solve the system:

$$3x - 4y + 4z = 7$$

$$x - y - 2z = 2$$

$$2x - 3y + 6z = 5$$

Example 2 continued

❖ Begin by representing the system as an augmented matrix:

$$\begin{pmatrix} 3 & -4 & 4 & | & 7 \\ 1 & -1 & -2 & | & 2 \\ 2 & -3 & 6 & | & 5 \end{pmatrix}$$

❖ Since the first number in the second row is a 1, we interchange rows 1 and 2 and leave row 3 unchanged:

$$\begin{pmatrix} 1 & -1 & -2 & | & 2 \\ 3 & -4 & 4 & | & 7 \\ 2 & -3 & 6 & | & 5 \end{pmatrix}$$

Continuation of example 2:

In this step, we will get zeros in the entries beneath the 1 in the first column: Multiply row 1 by -3 , add to row 2 and replace row 2; and -2*R1+R3 and replace R3:

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & -1 & 10 & 1 \end{array} \right)$$

To get a zero in the third row, second entry we multiply row 2 by -1 and add the result to R3 and replace R3 by that sum: Notice this operations "wipes out" row 3 so row consists entirely of zeros.

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This matrix corresponds to a dependent system with an infinite number of solutions:

Representation of a solution of a dependent system

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 2 \\ 0 & -1 & 10 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

We can interpret the second row of this matrix as **$-y + 10z = 1$**

Or **$10z - 1 = y$**

So, if we let **$z = t$** (arbitrary real number, then in terms of t,

$y = 10t - 1$.

Next we can express the variable x n terms of as follows: From the first row of the matrix, we have

$x - y - 2z = 2$. If **$z = t$** and

$y = 10t - 1$, we have

$x - (10t-1) - 2t = 2$ or $x - 12t + 1 = 2$; $x - 12t = 1$ or

$x = 12t + 1$

Our general solution can now be expressed in terms of t:

$(12t+1, 10t-1, t)$, where t is an arbitrary real number

REDUCED MATRIX

A matrix is a **REDUCED MATRIX** or is said to be in **REDUCED FORM** if

a)   each row consisting entirely of zeros is below any row having at least one nonzero element;

(b)   the left-most nonzero element in each row is 1;

(c)   all other elements in the column containing the left-most 1 of a given row are zeros;

(d)   the left-most 1 in any row is to the right of the left-most 1 in any row above.

GAUSS-JORDAN ELIMINATION

Step 1. Choose the leftmost nonzero column and use appropriate row operations to get a 1 at the top.

Step 2. Use multiples of the row containing the 1 from step 1 to get zeros in all remaining places in the column containing this 1.

Step 3. Repeat step 1 with the **SUBMATRIX** formed by (mentally) deleting the row used in step 2 and all rows above this row.

Step 4. Repeat step 2 with the **ENTIRE MATRIX**, including the mentally deleted rows. Continue this process until the entire matrix is in reduced form.

[*Note:* If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop before we find the reduced form, since we will have a contradiction: $0 = n$, $n \neq 0$. We can then conclude that the system has no solution

Solving a system using Gauss-Jordan Elimination

☞ Write the augmented matrix of system and apply all the steps of Gauss-Jordan elimination to bring the matrix into reduced form

☞ If at any point in this process we obtain a row with all zeros to the left of the vertical line and a nonzero number to the right, we can stop –

the system has no solution

3. If the number of leftmost 1's in reduced augmented coefficient matrix is equal to the number of variables in the system and there are no contradictions, then system is consistent (independent) and has a single (unique) solution.

4. If the number of leftmost 1's in reduced augmented coefficient matrix is less than the number of variables in the system and there are no contradictions, then system is consistent (dependent) and has a infinitely many solution.