Solution of the final examination Math 106, Semester I-1443

Question 1 :

a) $F'(x) = \cosh(\sin x).\cos x - 2x\cosh(x^2)$, **(1)**

$F'(0) = 1$ **(1)**.

b) $\displaystyle\int_1^2 \frac{dx}{x^2} = \frac{1}{2}.$ **(1)**

$\displaystyle\frac{1}{c^2} = \frac{1}{2} \Rightarrow c = \sqrt{2}$ since $-\sqrt{2} \notin [1,2].$ **(1)**

Question 2 :

a)

$$\int \frac{dx}{\sqrt{3^{2x}-1}} \overset{u=3^x}{=} \frac{1}{\ln 3}\int \frac{du}{u\sqrt{u^2-1}}. \qquad \textbf{(2)}$$

$$= \frac{1}{\ln 3}\sec^{-1}(3^x) + c. \qquad \textbf{(1)}$$

b)

$$\int \frac{\cot x}{\sqrt{1-\sin^2 x}}dx \overset{u=\sin x}{=} \int \frac{du}{u\sqrt{1-u^2}}. \qquad \textbf{(2)}$$

$$= -\text{sech}^{-1}(|\sin x|) + c. \qquad \textbf{(1)}$$

c) $Y = x^{\sin x}$, $\ln Y = \sin x \ln x$

$$\lim_{x\to 0^+} \ln Y = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}}$$

$$= \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} = 0. \qquad \textbf{(2.5)}$$

$\displaystyle\lim_{x\to 0^+} x^{\sin x} = 1.$ **(0.5)**

Question 3 :

a)

$$\int x\tan^{-1} x dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx \qquad (\mathbf{1.5})$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}dx$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c. \qquad (\mathbf{1.5})$$

b)

$$\int \frac{dx}{(1+x^2)^2} \overset{x=\tan\theta}{=} \int \frac{\sec^2\theta}{\sec^4\theta}d\theta$$
$$= \int \frac{1}{2} + \frac{1}{2}\cos(2\theta)d\theta$$
$$= \frac{\theta}{2} + \frac{1}{2}\sin\theta\cos\theta + c \qquad (\mathbf{2})$$
$$= \frac{1}{2}(\tan^{-1}x + \frac{x}{1+x^2}) + c. \qquad (\mathbf{1})$$

c)

$$\int \frac{dx}{\sqrt{x^2+8x+25}} \overset{u=x+4}{=} \int \frac{du}{\sqrt{9+u^2}} \qquad (\mathbf{2})$$
$$= \sinh^{-1}(\frac{x+4}{3}) + c. \qquad (\mathbf{1})$$

Question 4 :

a)

$$\int \frac{dx}{x^{\frac{1}{4}}+x^{\frac{1}{2}}} \overset{u=x^{\frac{1}{4}}}{=} 4\int \frac{u^3}{u+u^2}du$$

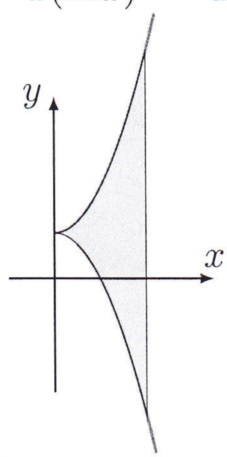
$$= 4\int (u-1+\frac{1}{1+u})du \qquad \textbf{(2)}$$

$$= 2x^{\frac{1}{2}} - 4x^{\frac{1}{4}} + 4\ln(1+x^{\frac{1}{4}}) + c. \qquad \textbf{(1)}$$

b) $\displaystyle\int_2^c \frac{dx}{x(\ln x)^3} \overset{u=\ln x}{=} \int_{\ln 2}^{\ln c} \frac{du}{u^3} = -\frac{1}{2}\left[\frac{1}{u^2}\right]_{\ln 2}^{\ln c}$ **(1.5)**

$\displaystyle\lim_{c\to+\infty} \int_2^c \frac{dx}{x(\ln x)^3} = \frac{1}{2(\ln 2)^2}$. So the integral converges and

$\displaystyle\int_2^{+\infty} \frac{dx}{x(\ln x)^3} = \frac{1}{2(\ln 2)^2}$. **(1.5)**



c) **(1)**

Intersection points $x^2+1 = 1-x^2 \Rightarrow x = 0$.

$$A = \int_0^2 2x^2 dx = \frac{16}{3}. \qquad \textbf{(2)}$$

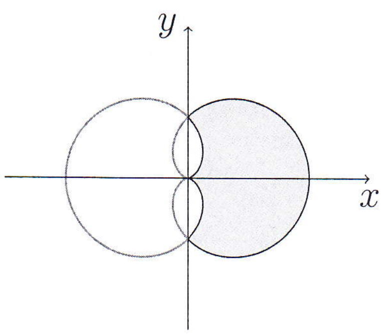
Question 5 :

a) $4 - x^2 = 0 \iff x = \pm 2.$

$$\begin{aligned} V &= \int_0^2 2\pi x(4 - x^2)dx \qquad (\mathbf{2}) \\ &= 2\pi \left[2x^2 - \frac{1}{4}x^4\right]_0^2 = 8\pi. \qquad (\mathbf{1}) \end{aligned}$$

b)

$$\begin{aligned} S &= 2\pi \int_0^1 t^3\sqrt{4 + 9t^4}dt \qquad (\mathbf{1.5}) \\ &= \frac{\pi}{18}\int_4^{13} \sqrt{u}du = \frac{\pi}{27}(13^{\frac{3}{2}} - 8). \qquad (\mathbf{1.5}) \end{aligned}$$

c) 

(1)

$$\begin{aligned} A &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos\theta)^2 - (1 - \cos\theta)^2 d\theta \qquad (\mathbf{1}) \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cos\theta d\theta = 4. \qquad (\mathbf{1}) \end{aligned}$$