Lecture: Fourier Series

1. Periodic Functions

Definition (Periodic Function).

A function $f : \mathbb{R} \to \mathbb{R}$ is called **periodic** with period $T > 0$ if

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

The smallest positive number T satisfying this property is called the **fundamental period** of f.

Examples.

- The function $f(x) = \sin x$ is periodic with fundamental period 2π.
- The functions
$$\cos\left(\frac{n\pi x}{L}\right), \qquad \sin\left(\frac{n\pi x}{L}\right)$$
are periodic with period $\frac{2L}{n}$.

How to plot a periodic function.

- Plot the function on one basic interval, for example $[0, T]$ or $[-L, L]$.
- Repeat this graph to the right and left by shifting it by multiples of the period T.
- That is, copy the graph using the rule
$$f(x+T) = f(x), \quad f(x-T) = f(x).$$

Example.
If $f(x) = x$ on $[0, 2\pi]$, then its periodic extension is obtained by repeating this line segment every 2π units.

Remark. A periodic function is completely determined by its values on any interval of length T.

2. Fourier Series Representation

Fourier Series Representation.

Let f be a $2L$-periodic function. Then f can be represented by its Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

at all points where the series converges.
The coefficients of the Fourier series are given by:

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1.$$

4. Orthogonality

Orthogonality of Trigonometric Functions.

For integers $m, n \geq 1$, the following orthogonality relations hold on $[-L, L]$:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} L, & m = n, \\ 0, & m \neq n, \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} L, & m = n, \\ 0, & m \neq n. \end{cases}$$

Remark. These relations show that sine and cosine functions behave like orthogonal vectors. This property allows us to compute the Fourier coefficients.

Even and Odd Periodic Functions

Even and Odd Symmetry.

A function is:

- **Even** if $f(-x) = f(x)$ (symmetry about the y-axis),
- **Odd** if $f(-x) = -f(x)$ (symmetry about the origin).

Periodic extension.

- First construct the function on $[-L, L]$ using symmetry.
- Then extend periodically with period $2L$.

Examples.

$$\text{Even: } |x|,\ x^2,\ \cos x,\ 1 \qquad \text{Odd: } x,\ x^3,\ \sin x,\ \text{sign}(x)$$

Fourier Series Simplification.

Even case:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \qquad b_n = 0,$$

with

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx.$$

Odd case:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \qquad a_n = 0,$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

- Even symmetry $\Rightarrow$ cosine series.
- Odd symmetry $\Rightarrow$ sine series.
- Computations reduce to integrals on $[0, L]$.

Remark. Recognizing symmetry greatly simplifies Fourier series computations.

7. Convergence (Dirichlet Theorem)

Dirichlet Theorem (Convergence of Fourier Series).
A function f is **piecewise** C^1 if:

- f is continuously differentiable on each subinterval,
- and has only finitely many discontinuities (jump points).

For every $x \in \mathbb{R}$, the Fourier series of f satisfies

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = \frac{f(x^-) + f(x^+)}{2},$$

where

$$f(x^-) = \lim_{t \to x^-} f(t), \qquad f(x^+) = \lim_{t \to x^+} f(t).$$

- If f is **continuous** at x, then

  $$f(x^-) = f(x^+) = f(x),$$

  and the Fourier series converges to $f(x)$.

- If f has a **jump discontinuity** at x, then

  $$f(x^-) \neq f(x^+),$$

  and the Fourier series converges to the **midpoint of the jump**:

  $$\frac{f(x^-) + f(x^+)}{2}.$$

At discontinuities, the Fourier series does not converge to $f(x)$ itself, but to the **average of the left and right limits**.

8. Example 1

Consider the function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \end{cases}$$

extended periodically with period 2π.

The function is odd, hence its Fourier series contains only sine terms:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \qquad a_0 = 0, \quad a_n = 0.$$

The coefficients are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx.$$

A direct computation gives

$$\int_0^{\pi} \sin(nx)\, dx = \frac{1 - (-1)^n}{n},$$

so that

$$b_n = \frac{2}{\pi} \cdot \frac{1 - (-1)^n}{n}.$$

This implies

$$b_n = \begin{cases} \dfrac{4}{\pi n}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Therefore, the Fourier series of f is

$$f(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right).$$

By Dirichlet's theorem, the series converges pointwise and satisfies

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(nx) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \\ 0, & x = 0, \ \pm\pi. \end{cases}$$

At the points of discontinuity $x = 0$ and $x = \pm\pi$, the series converges to the midpoint:

$$\frac{1 + (-1)}{2} = 0.$$

$$\boxed{\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin(nx) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0, \\ 0, & x = 0, \ \pm\pi. \end{cases}}$$

9. Example 2

Consider the function

$$f(x) = x, \qquad -\pi < x < \pi,$$

extended periodically with period 2π.

The function is odd, so its Fourier series contains only sine terms:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \qquad a_0 = 0, \quad a_n = 0.$$

The coefficients are given by

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx.$$

Using integration by parts:

$$\int x \sin(nx)\, dx = -\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}.$$

Evaluating from 0 to π:

$$\int_0^{\pi} x \sin(nx)\, dx = -\frac{\pi(-1)^n}{n}.$$

Hence

$$b_n = -2\frac{(-1)^n}{n}.$$

Therefore, the Fourier series is

$$x = \sum_{n=1}^{\infty} \left(-2\frac{(-1)^n}{n} \sin(nx) \right), \qquad -\pi < x < \pi.$$

Equivalently,

$$x = 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \right).$$

At $x = \pm\pi$, the series converges to the midpoint:

$$\frac{\pi + (-\pi)}{2} = 0.$$

10. Example 3: Absolute Value

Consider the function

$$f(x) = |x|, \qquad -\pi \le x \le \pi,$$

extended periodically with period 2π.

The function is even, so its Fourier series contains only cosine terms:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \qquad b_n = 0.$$

We compute

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi, \qquad \frac{a_0}{2} = \frac{\pi}{2}.$$

The coefficients are

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx.$$

Using integration by parts:

$$\int x\cos(nx)\,dx = \frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}.$$

Evaluating from 0 to π:

$$\int_0^{\pi} x\cos(nx)\,dx = \frac{(-1)^n - 1}{n^2}.$$

Thus

$$a_n = \frac{2}{\pi}\cdot\frac{(-1)^n - 1}{n^2}.$$

Hence

$$a_n = \begin{cases} -\dfrac{4}{\pi n^2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Therefore,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\cos(nx)}{n^2}, \qquad -\pi < x < \pi.$$

At every point, the series converges to $|x|$.

Setting $x = 0$ gives

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

11. Example

Consider the function

$$f(x) = x + 1, \qquad -\pi < x < \pi,$$

extended periodically with period 2π.

This function is **neither even nor odd**, since

$$f(-x) = -x + 1 \neq f(x), \qquad f(-x) \neq -f(x).$$

Therefore, its Fourier series contains **both cosine and sine terms**:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

The coefficients are given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+1)\, dx,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+1) \cos(nx)\, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x+1) \sin(nx)\, dx.$$

Simplification using symmetry.

- The function can be written as

$$f(x) = x + 1.$$

- The term x is odd $\Rightarrow$ contributes only to sine terms.
- The constant 1 is even $\Rightarrow$ contributes only to cosine terms.

Final decomposition.

$$f(x) = \underbrace{1}_{\text{cosine part}} + \underbrace{x}_{\text{sine part}}.$$

Thus,

$$f(x) = 1 + \sum_{n=1}^{\infty} \left(-2 \frac{(-1)^n}{n} \sin(nx) \right).$$

Key idea.
If a function is neither even nor odd:

- both cosine and sine terms appear,
- but it can often be decomposed into an even part and an odd part:

$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x).$$

$$f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}, \qquad f_{\text{odd}}(x) = \frac{f(x) - f(-x)}{2}.$$

12. Half-Range Expansions

Half-Range Expansions.

Suppose a function f is defined only on the interval $(0, L)$. We can extend it to $[-L, L]$ in two natural ways:

Even extension (Cosine series).

- Extend f so that $f(-x) = f(x)$.
- The resulting function is even.
- Its Fourier series contains only cosine terms.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

Odd extension (Sine series).

- Extend f so that $f(-x) = -f(x)$.
- The resulting function is odd.
- Its Fourier series contains only sine terms.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

- Cosine series $\Rightarrow$ even extension.
- Sine series $\Rightarrow$ odd extension.
- We only integrate on $(0, L)$.

Example: Half-Range Sine Series

Consider the function

$$f(x) = x, \qquad 0 < x < L.$$

We construct its **odd extension** to $[-L, L]$, so the Fourier series will be a sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

The coefficients are

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

Using integration by parts:

$$u = x, \qquad dv = \sin\left(\frac{n\pi x}{L}\right) dx,$$

$$v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right).$$

Then

$$\int x \sin\left(\frac{n\pi x}{L}\right)\, dx = -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right) dx.$$

$$= -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right).$$

Evaluating from 0 to L:

$$\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\, dx = -\frac{L^2}{n\pi}(-1)^n.$$

Thus

$$b_n = \frac{2}{L}\cdot\left(-\frac{L^2}{n\pi}(-1)^n\right) = -\frac{2L}{n\pi}(-1)^n.$$

Therefore,

$$x = \sum_{n=1}^{\infty}\left(-\frac{2L}{n\pi}(-1)^n\right)\sin\left(\frac{n\pi x}{L}\right), \qquad 0 < x < L.$$

Remark. This represents the odd periodic extension of x on $(0, L)$.