Motivation.

Fourier series represent **periodic** functions as sums of sines and cosines. But many functions are defined on $(-\infty, \infty)$ and are **not periodic**. In such cases, we use the **Fourier integral** to represent functions using **integrals** instead of infinite sums.

Fourier Integral Representation.

Let $f(x)$ be real-valued on $(-\infty, \infty)$. Assume:

(i) f and f' are piecewise continuous on every bounded interval,

(ii) f is absolutely integrable:

$$\int_{-\infty}^{\infty} |f(x)|\, dx \quad \text{converges.}$$

Then f has the Fourier integral representation

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \Big[A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x) \Big]\, d\lambda,$$

where

$$A(\lambda) = \int_{-\infty}^{\infty} f(t)\cos(\lambda t)\, dt, \qquad B(\lambda) = \int_{-\infty}^{\infty} f(t)\sin(\lambda t)\, dt.$$

Convergence rule.

If f is continuous at x, then the Fourier integral converges to $f(x)$.

If $x = x_0$ is a point of discontinuity, it converges to the midpoint value:

$$\frac{1}{2}\Big(f(x_0^-) + f(x_0^+) \Big) = \frac{1}{\pi} \int_0^{\infty} \Big[A(\lambda)\cos(\lambda x_0) + B(\lambda)\sin(\lambda x_0) \Big]\, d\lambda.$$

Symmetry.

- If f is **even** $(f(-x) = f(x))$, then $B(\lambda) = 0$ and

$$f(x) = \frac{1}{\pi} \int_0^{\infty} A(\lambda)\cos(\lambda x)\, d\lambda, \qquad A(\lambda) = 2\int_0^{\infty} f(t)\cos(\lambda t)\, dt.$$

- If f is **odd** $(f(-x) = -f(x))$, then $A(\lambda) = 0$ and

$$f(x) = \frac{1}{\pi} \int_0^{\infty} B(\lambda)\sin(\lambda x)\, d\lambda, \qquad B(\lambda) = 2\int_0^{\infty} f(t)\sin(\lambda t)\, dt.$$

Examples

Example 1.

Define

$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Since f is even, $B(\lambda) = 0$. Then

$$A(\lambda) = 2\int_0^1 \cos(\lambda x)\, dx = \frac{2\sin\lambda}{\lambda}.$$

Hence

$$f(x) = \frac{1}{\pi}\int_0^\infty A(\lambda)\cos(\lambda x)\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos(\lambda x)\, d\lambda.$$

At $x = 0$,

$$1 = f(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda \quad \Rightarrow \quad \int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda = \frac{\pi}{2}.$$

Example 2.

Define

$$f(x) = \begin{cases} -1, & -1 \le x \le 0, \\ 1, & 0 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

Since f is odd, $A(\lambda) = 0$. Then

$$B(\lambda) = 2\int_0^1 \sin(\lambda x)\, dx = \frac{2(1-\cos\lambda)}{\lambda}.$$

Hence

$$f(x) = \frac{1}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)\, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda}\sin(\lambda x)\, d\lambda.$$

Example 3.

Let

$$f(x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Since f is even, $B(\lambda) = 0$ and

$$A(\lambda) = 2 \int_0^1 (1 - t) \cos(\lambda t) \, dt.$$

Compute

$$A(\lambda) = \frac{2(1 - \cos \lambda)}{\lambda^2}.$$

Therefore

$$\boxed{f(x) = \frac{1}{\pi} \int_0^\infty \frac{2(1 - \cos \lambda)}{\lambda^2} \cos(\lambda x) \, d\lambda}.$$

At $x = 0$, $f(0) = 1$, so

$$1 = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \lambda}{\lambda^2} \, d\lambda \quad \Rightarrow \quad \boxed{\int_0^\infty \frac{1 - \cos \lambda}{\lambda^2} \, d\lambda = \frac{\pi}{2}.}$$

Example 4.

Let

$$f(x) = \begin{cases} 1, & -1 \le x < 0, \\ 0, & 0 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

Here

$$A(\lambda) = \int_{-1}^{0} \cos(\lambda t)\,dt = \frac{\sin\lambda}{\lambda}, \qquad B(\lambda) = \int_{-1}^{0} \sin(\lambda t)\,dt = \frac{1-\cos\lambda}{\lambda}.$$

Hence

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda}{\lambda}\cos(\lambda x) + \frac{1-\cos\lambda}{\lambda}\sin(\lambda x)\right] d\lambda}.$$

At $x = 0$, $\sin(\lambda x) = 0$ and $\cos(\lambda x) = 1$, so

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}.$$

Also, by midpoint rule:

$$\frac{1}{2}\big(f(0^-) + f(0^+)\big) = \frac{1}{2}(1+0) = \frac{1}{2}.$$

$$\boxed{f(0) = \frac{1}{2}}.$$

Exercise.

Show that

$$\int_0^\infty \frac{\sin(\pi\lambda)}{1-\lambda^2}\sin(\lambda x)\,d\lambda = \begin{cases} \frac{\pi}{2}\sin x, & 0 \le x \le \pi, \\ 0, & x > \pi, \end{cases}$$

(and the value is 0 also for $x < 0$).

Hint: Use the Fourier **sine** transform.

Solution (sketch).

Consider

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi, \\ 0, & x > \pi, \end{cases} \qquad \text{(and } f(x) = 0 \text{ for } x < 0\text{).}$$

Its sine transform is

$$B(\lambda) = \int_0^\infty f(t)\sin(\lambda t)\,dt = \int_0^\pi \sin t\,\sin(\lambda t)\,dt.$$

Using product-to-sum,

$$\sin t\,\sin(\lambda t) = \tfrac{1}{2}\Big(\cos((\lambda-1)t) - \cos((\lambda+1)t)\Big),$$

so

$$B(\lambda) = \frac{1}{2}\left[\frac{\sin((\lambda-1)\pi)}{\lambda-1} - \frac{\sin((\lambda+1)\pi)}{\lambda+1}\right] = \frac{\sin(\pi\lambda)}{1-\lambda^2}.$$

The inversion formula gives for $x > 0$:

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\lambda)\sin(\lambda x)\,d\lambda,$$

hence

$$\int_0^\infty \frac{\sin(\pi\lambda)}{1-\lambda^2}\sin(\lambda x)\,d\lambda = \frac{\pi}{2}\,f(x) = \begin{cases} \dfrac{\pi}{2}\sin x, & 0 \le x \le \pi, \\ 0, & x > \pi, \end{cases}$$

and it is 0 for $x < 0$.