Fist Midterm Exam Solutions for math 203 (1st semester 1446)

1. Find the sum of the series $\sum_{n=1}^{\infty} \frac{4}{7^{n-1}} + \sum_{n=4}^{\infty} \frac{12}{(2n-6)(2n-4)}$.

Solution:

$$A = \sum_{n=1}^{\infty} \frac{4}{7^{n-1}} = 4 \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^{n-1} = 4 \sum_{n=0}^{\infty} \left(\frac{1}{7}\right)^{n} 4 \left(\frac{1}{1-\frac{1}{7}}\right) = \frac{14}{3} .$$

$$B = \sum_{n=4}^{\infty} \frac{12}{(2n-6)(2n-4)} = 3 \sum_{n=4}^{\infty} b_n$$

$$3b_n = \frac{12}{(2n-6)(2n-4)} = \frac{3}{(n-3)(n-2)} = 3 \left(\frac{1}{n-3} - \frac{1}{n-2}\right)$$

$$S_n = b_4 + \ldots\ b_{n+3}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$$

Hence

$$B = \sum_{n=4}^{\infty} \frac{12}{(2n-6)(2n-4)} = 3 \sum_{n=4}^{\infty} \left(\frac{1}{n-3} - \frac{1}{n-2}\right) = 3$$

Therefore

$$\sum_{n=1}^{\infty} \frac{4}{7^{n-1}} + \sum_{n=4}^{\infty} \frac{12}{(2n-6)(2n-4)} = A + B = \frac{14}{3} + 3 = \frac{23}{3}$$

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{|\cos n|}{\sqrt{3+n^2+2n^3}}$ is convergent or divergent.

Solution:

$$a_n = \frac{|\cos n|}{\sqrt{3+n^2+2n^3}} \leq \frac{1}{\sqrt{3+n^2+2n^3}} \leq \frac{1}{\sqrt{n^3}} = \frac{1}{n^{\frac{3}{2}}}$$

Now, the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges. that is, since it is a p – series with $p = \frac{3}{2} > 1$. And by the basic comparison test the series

$\sum_{n=1}^{\infty} \frac{|\cos n|}{\sqrt{3+n^2+2n^3}}$ is convergent.

3. Find the radius and the interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{n\,(Ln(n))^2}$$

Solution:

Taking $u_n = \dfrac{(x-1)^n}{n\ (Ln(n))^2}$ then

$$\left|\frac{u_{n+1}}{u_n}\right| = |x-1| \frac{n\,(Ln(n))^2}{(n+1)\,(Ln(n+1))^2} .$$

Now taking $f(t) = \frac{Ln(t)}{Ln(t+1)}$ *then* $f(n) = \frac{Ln(n)}{Ln(n+1)}$ and

$$\lim_{t\to\infty} f(t) = \underset{t\to\infty}{Lim} \frac{Ln(t)}{Ln(t+1)} \overset{L'Hopital}{=} \underset{t\to\infty}{Lim} \frac{t+1}{t} = 1.$$

So

$$\underset{n\to\infty}{Lim} \frac{(Ln(n))^2}{(Ln(n+1))^2} = 1 \text{ and since } \underset{n\to\infty}{Lim} \frac{n}{n+1} = 1$$

We have

$$\underset{n\to\infty}{Lim} \left|\frac{u_{n+1}}{u_n}\right| = |x-1|$$

By the ratio test, the series converges if $|x-1| < 1$

That is $-1 < x-1 < 1 \Rightarrow 0 < x < 2$.

If $x = 2$ we have the series $\sum_{n=2}^{\infty} \frac{1}{n\,(Ln(n))^2} = \sum_{n=1}^{\infty} \mathrm{a_n}$

$f(x) = \frac{1}{x\,(Ln(x))^2}$ is decreasing continuous function on $[2\,,\infty]$ and

$f(n) = a_n$. Also

$$\int_2^{\infty} \frac{dx}{x\,(Ln(x))^2} = \frac{1}{Ln2}$$

Hence by the integral test the series $\sum_{n=2}^{\infty} \frac{1}{n\,(Ln(n))^2}$ converges.

If $x = 0$ we get the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(Ln(n))^2} = \sum_{n=2}^{\infty} \mathrm{b}n$.

But $\sum_{n=2}^{\infty} |\mathrm{b_n}| = \sum_{n=2}^{\infty} \frac{1}{n\,(Ln(n))^2}$ which converges, therefore $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(Ln(n))^2}$ converges. So

The interval of convergence is $I = [\,0\,,2\,]$

and the radius of convergence is $r = \frac{2}{2} = 1$.

4. Find the power series representation for function $f(x)=\dfrac{1}{1+x^2}$

if $-1<x<1$ and use it to find power series representation of

$tan^{-1}(x)$, $if\ -1\le x\le 1$.

Solution:

We know that $\sum\limits_{n=0}^{\infty}u^n=\dfrac{1}{1-u} \Leftrightarrow |u|<1$, taking $u=-x^2$ we get

$\dfrac{1}{1+x^2}=\sum\limits_{n=0}^{\infty}(-1)^n x^{2n} \Leftrightarrow |x|<1$, integrating both sides, we get for any

$x\in(-1\,,1\,)$:

$$\int_0^x \frac{1}{1+t^2}\,dt=\int_0^x\sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt$$

Therefore

$$\int_0^x \frac{1}{1+t^2}\,dt=\tan^{-1}(x)=\int_0^x\sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

If $x=-1$ we have the series $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^{n+1}}{2n+1}$ and If $x=1$ we get the series

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, both of these series converge by the alternating series test.

And we conclude that:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad if \quad -1 \le x \le 1.$$

5. Find the Maclaurin series for function $f(x) = e^x$ and approximate the integral

$$\int_0^{0.1} x^2 e^{-x^2} \, dx$$

using the first three nonzero terms.

Solution:

Maclaurin series for a function f is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$,

if $f(x) = e^x$ then $f^{(n)}(x) = e^x$ and $f^{(n)}(0) = 1$. Hence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} .$$

Replacing x by $-x^2$ we get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

And

$$x^2 e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

Therefore

$$\int_0^{0.1} x^2 e^{-x^2} dx = \int_0^{0.1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} dx$$

$$\approx \left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} \right]_0^{0.1} = \frac{(0.1)^3}{3} - \frac{(0.1)^5}{5} + \frac{(0.1)^7}{14}$$

$$\approx 0.00033134$$