FINAL EXAMINATION, SEMESTER I, 2022
Dept. Math., College of Science, KSU
Math: 107 Full Mark: 40 Time: 3 Hours

Q1. [Marks: 3+3+3=9]

(a) Find the values of λ for which the following system of equations has a unique solution:

$$\begin{aligned} 2x_1 + 3x_2 + \quad x_3 &= -1 \\ x_1 + 2x_2 + \quad x_3 &= 0 \\ 3x_1 + x_2 + (\lambda^2 - 6)x_3 &= \lambda - 3. \end{aligned}$$

(b) Let A be a square matrix with $\det(A) = 1$ and

$$adj(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute A^{-1} and deduce A.

(c) By using Cramer's Rule solve the system of equations:

$$\begin{aligned} -p + q - 2r &= 1 \\ p - q + 9r &= -2 \\ 5q + r &= 4. \end{aligned}$$

Q2. [Marks: 3+3+3=9]

(a) Let $\mathbf{u} = \langle 1, 2, -1 \rangle$, $\mathbf{v} = \langle 0, 1, -1 \rangle$ and $\mathbf{w} = \langle 2, 3, 1 \rangle$. (i) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

(ii) Find the component of $\mathbf{u} + 2\mathbf{v}$ along $\mathbf{w}$.

(b) Find an equation of the plane that contains the point $P(4, -3, 0)$ and the line: $x = t + 5$, $y = 2t - 1$, $z = -t + 7$.

(c) Identify the surface $x - y^2 - z^2 = 0$, give traces, and sketch.

Q3. [Mark: 6+3+3=12]

(a) The position vector of a moving point at time t is $\mathbf{r}(t) = \langle 4\cos t, 9\sin t, t \rangle$. Find the tangential and normal components of acceleration, and curvature at time t.

(b) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+3y^2}{x^2+y^2}$ does not exist.

(c) The temperature T at (x, y) is given by $T = 5(x^2 + y^2)^2$, where T is in degrees and x and y are in centimeters. Use differential to approximate the temperature difference between points $(1, 1)$ and $(1.01, 0.98)$.

Q4. [Marks: 3+3+4=10]

(a) Use partial derivative to find $\frac{\partial z}{\partial x}$ if $(x^2 + 1)z^3 - y^2 z^2 + xyz = 3$.

(b) Let $f(x, y, z) = xyz$. Find the directional derivative of f at the point $A(1, 1, 1)$ in the direction of the vector $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(c) Let $g(x, y) = x^2 + y^2 + xy^2 + 9$. Find the local extrema and saddle point if any.

Solution M 107 Final Exam

Q1 (a) $\left(\begin{array}{ccc|c} 2 & 3 & 1 & -1 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & \lambda^2-6 & \lambda-3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & \lambda^2-4 & \lambda-2 \end{array}\right)$

Hence the system has a unique solution for all $\lambda \in \mathbb{R} \setminus \{-2, 2\}$.

Mark 3

(b) $\det(A) = 1$ and $\operatorname{adj}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. So,

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ Hence,}$$

$$A = (A^{-1})^{-1} = (\operatorname{adj}(A))^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Mark: 1.5 + 1.5 = 3

(c) $\det(A) = \begin{vmatrix} -1 & 1 & -2 \\ 1 & -1 & 9 \\ 0 & 5 & 1 \end{vmatrix} = 35$

- $\det(A_1) = \begin{vmatrix} 1 & 1 & -2 \\ -2 & -1 & 9 \\ 4 & 5 & 1 \end{vmatrix} = 4$ Mark 1
- $\det(A_2) = \begin{vmatrix} -1 & 1 & -2 \\ 1 & -2 & 9 \\ 0 & 4 & 1 \end{vmatrix} = 29$ Mark 1
- $\det(A_3) = \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 5 & 4 \end{vmatrix} = -5$ Mark 1

Thus, $p = 4/35$, $q = 29/35$, and $r = -5/35 = -1/7$

Q_2 (a)(i) $\|\underline{u}\| = \sqrt{6}$, $\|\underline{v}\| = \sqrt{2}$ Mark 1.5

$\underline{u}\cdot\underline{v} = 3$ $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\|\underline{v}\|} = \dfrac{3}{\sqrt{6}\sqrt{2}} = \dfrac{3}{\sqrt{12}} = \dfrac{3}{2\sqrt{3}}$

(ii) $\underline{u} + 2\underline{v} = \langle 1, 4, -3\rangle$ Mark 1.5

$$\text{comp}_{\underline{w}}\, \underline{u}+2\underline{v} = \frac{\langle 1,4,-3\rangle\cdot\langle 2,3,1\rangle}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

(b) The point $Q(5,-1,7)$ with $t=0$ and $R(6,1,6)$ with $t=1$ lie on the line and in the plane. As $P(4,-3,0)$ we get $\vec{QP} = \langle -1,-2,-7\rangle$ and $\vec{QR} = \langle 1,2,-1\rangle$.

Thus,
$$\vec{QP}\times\vec{QR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & -7 \\ 1 & 2 & -1 \end{vmatrix} = \langle 16, -8, 0\rangle$$ Mark 2

Therefore, the equation of the plane is
$$16(x-4) - 8(y+3) + 0(z-0) = 0$$
or, $16x - 8y - 64 - 24 = 0$ Mark 1
i.e. $2x - y = 11$

(c) xy-trace : $x = y^2$ parabola
xz-trace : $x = z^2$ parabola
parallel to xy-plane $(x=1)$: $y^2+z^2=1$ circle.
The surface $x = y^2 + z^2$ is a paraboloid.

Marks $1+1+1=3$

Q_3 (a) $\underline{r}(t) = \langle 4\cos t, 9\sin t, t \rangle$

$\underline{r}'(t) = \langle -4\sin t, 9\cos t, 1 \rangle$

$\underline{r}''(t) = \langle -4\cos t, -9\sin t, 0 \rangle$

$$\underline{r}'(t)\cdot\underline{r}''(t) = 16\sin t\cos t - 81\sin t\cos t$$
$$= -65\sin t\cos t$$

$$\|\underline{r}'(t)\| = \sqrt{16\sin^2 t + 81\cos^2 t + 1}$$

Mark 2

$$a_T = \frac{-65\sin t\cos t}{(16\sin^2 t + 81\cos^2 t + 1)^{\frac{1}{2}}}$$

$$a_N = \frac{(81\sin^2 t + 16\cos^2 t + 1296)^{\frac{1}{2}}}{(16\sin^2 t + 81\cos^2 t + 1)^{1/2}}$$

Mark 3

$$k = a_N \cdot \frac{1}{16\sin^2 t + 81\cos^2 t + 1}$$
$$= \frac{(81\sin^2 t + 16\cos^2 t + 1296)^{\frac{1}{2}}}{(16\sin^2 t + 81\cos^2 t + 1)^{3/2}}$$

Mark 1

(b) • $x \neq 0$, $f(x,0) = 1$, $\lim_{x\to 0} \frac{x^2}{x^2} = 1$

• $y \neq 0$, $f(0,y) = 3$, $\lim_{y\to 0} \frac{3y^2}{y^2} = 3$

• By Two-path rule $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

Marks $1+1+1 = 3$

Q_3 (c) $T(x,y) = 5(x^2+y^2)^2$,
$dx = 0.01$, $dy = -0.02$.

Mark 1

$$dT = 10(x^2+y^2)(2x)\,dx + 10(x^2+y^2)(2y)\,dy$$
$$= 20x(x^2+y^2)\,dx + 20y(x^2+y^2)\,dy$$

Mark 1

$(x,y) = (1,1)$,

$$dT = 40(.01 - .02) = -0.4$$

Mark 1

Q_4 (a) $F(x,y,z) = (x^2+1)z^3 - y^2z^2 + xyz - 3$

$$\frac{\partial z}{\partial x} = -\frac{(2xz^3 + yz)}{3(x^2+1)z^2 - 2y^2z + xy}.$$

Mark: 1.5 + 1.5 = 3

(b) $f(x,y,z) = xyz$

Mark 1

$$\underline{u} = \frac{\langle 1,1,1\rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$\nabla f = \langle yz, xz, xy \rangle$$

Mark 1

$$D_{\underline{u}} f(x,y,z) = \nabla f \cdot \underline{u}\,(x,y,z)$$
$$= \langle 1,1,1\rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$
$$= \frac{3}{\sqrt{3}} = \sqrt{3}$$

Mark 1

Q4 (c) $g(x,y) = x^2 + y^2 + xy^2 + 9$

$$\Rightarrow \left.\begin{array}{l} g_x = 2x + y^2 = 0 \\ g_y = 2y + 2xy = 0 \end{array}\right\} \Rightarrow$$

$$2y - 2\cdot\tfrac{1}{2}y^2 y = 0 \Rightarrow$$
$$2y - y^3 = 0 \Rightarrow y = 0,\ y = \pm\sqrt{2}$$

Mark 1

so, $\begin{cases} x = 0, y = 0 \\ x = -1, y = \sqrt{2} \\ x = -1, y = -\sqrt{2} \end{cases}$

Critical points are : $(0,0)$, $(-1,\sqrt{2})$ and $(-1,-\sqrt{2})$.

$$D(x,y) = \begin{vmatrix} 2 & 2y \\ 2y & 2+2x \end{vmatrix}$$

Mark 1

- $D(0,0) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$, $g_{xx} > 0$
  $\Rightarrow$ local min
- $D(-1,\sqrt{2}) = \begin{vmatrix} 2 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{vmatrix} = -8 < 0 \Rightarrow$ saddle point
- $D(-1,-\sqrt{2}) = \begin{vmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{vmatrix} = -8 < 0$
  $\Rightarrow$ saddle point

Mark 2