King Saud University: Mathematics Department Math-254
Third Semester 1444 H Final Examination Solution
Maximum Marks = 40 Time: 180 mins.

Name of the Student:———————————— **I.D. No.** ——————————-

Name of the Teacher:———————————— **Section No.** ——————————

Note: Check the total number of pages are Six (6).

(**15 Multiple choice questions and Two (2) Full questions**)

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	a	b	c	b	c	a	c	a	b	b

Q. No.	11	12	13	14	15
a,b,c,d	a	a	c	c	b

Quest. No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 15		30
Q. 16		5
Q. 17		5
Total		40

Question 1: The number of bisections required to solve the equation $x^3 - 2x = 1$ in $[1.5, 2]$ accurate to within 10^{-4} is:

(a) 11 (b) 13 (c) 15 (d) None of these

Question 2: Given $x_0 = 0$ and $x_1 = 0.1$, then the next approximation x_2 of the solution of the reciprocal of 5 using the Secant method is:

(a) 0.175 (b) 0.15 (c) 0.1 (d) None of these

Question 3: The order of convergence of the Newton's method for $f(x) = \tan x$ at the root $\alpha = \pi$ is:

(a) 3 (b) 1 (c) 2 (d) None of these

Question 4: The l_∞-norm of the inverse of the Jacobian matrix of the nonlinear system $x^2 + y^2 = 1,\ xy = 1$ at the point $(1, 0)$ is:

(a) 0.5 (b) 2 (c) 1 (d) None of these

Question 5: In the LU factorization with Doolittles method of the matrix $A = \begin{pmatrix} 1 & -1 \\ \alpha & 1 \end{pmatrix}$, the matrix U is singular if α is equal to:

(a) -1 (b) 1 (c) ± 1 (d) None of these

Question 6: The first approximation for solving linear system $A\mathbf{x} = [1, 3]^T$ using Jacobi iterative method wit $A = \begin{pmatrix} -4 & 5 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{x}^{(0)} = [0.5, 0.5]^T$ is:

(a) $[1.375, 1.315]^T$ (b) $[0.375, 1.250]^T$ (c) $[1.375, 1.250]^T$ (d) None of these

Question 7: Solving linear system $A\mathbf{x} = [4, 5]^T$, with $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, by Gauss-Seidel iterative method, if $\|\mathbf{x}^{(1)} - \mathbf{x}^{(0)}\| = 0.75$, then the number of iterations needed to get an accuracy within 10^{-2} is:

(a) 6 (b) 8 (c) 10 (d) None of these

Question 8: If $\hat{x} = [1.01, 0.99]^T$ is an approximate solution for the system of two linear equations $2x - y = 1$ and $x + y = 2$, then the error bound for the relative error is:

(a) 0.025 (b) 0.035 (c) 0.045 (d) None of these

Question 9: Using data points: $(0, -2), (0.1, -1), (0.15, 1), (0.2, 2), (0.3, 3)$, the best approximate value of $f(0.25)$ by a linear Lagrange polynomial is:

(a) 2.5 (b) 1.5 (c) 3.5 (d) None of these

Question 10: If $f(x) = x^2e^x$, then $f[1, 1, 2]$ equals to:

(a) $4e^2 - 4e$ (b) $4e^2 + 4e$ (c) $4e^2 - 3e$ (d) None of these

Question 11: Using data points: $(0, -2), (0.1, -1), (0.15, 1), (0.2, 2), (0.3, 3)$, the best approximation of $f'(0.25)$ using 3-point difference formula is:

(a) 15.0 (b) 20.0 (c) 10.0 (d) None of these

Question 12: Using data points: $(0, -2), (0.1, -1), (0.15, 1), (0.2, 2), (0.3, 3)$, then the best approximation of $f''(0.15)$ using 3-point difference formula is:

(a) -44.44 (b) -400.00 (c) -3.33 (d) None of these

Question 13: Using data points: $(0, -2), (0.1, -1), (0.15, 1), (0.2, 2), (0.3, 3)$, the best approximate value of the integral $\int_0^{0.3} f(x)\,dx$, using the composite Trapezoidal rule is:

(a) 0.1 (b) 0.15 (c) 0.25 (d) None of these

Question 14: If $f(0) = 3, f(1) = \frac{\alpha}{2}, f(2) = \alpha$, and the Simpson's rule for $\int_0^2 f(x)\,dx = 4$, then the value of α is:

(a) 3.0 (b) 2.0 (c) 1.5 (d) None of these

Question 15: Given $xy' + y = 1$, $y(1) = 0$, the approximate value of $y(2)$ using Euler's method when $n = 1$ is:

(a) 1.0 (b) 1.5 (c) 2.0 (d) None of these

Question 16: Use the following table to find the best approximation of $f(0.6)$ by using quadratic Lagrange interpolating polynomial for equally spaced data points:

x	0.15	0.2	0.3	0.5	0.55	0.8	1
$f(x)$	-0.0427	-0.0644	-0.1084	-0.1733	-0.1808	-0.1428	0.0000

The function tabulated is $f(x) = x^2 \ln x$. Compute the absolute error and an error bound (using error bound formula for equally spaced data points) for the approximation.

Solution. Given $x = 0.6$, so, the best three points for the quadratic Lagrange interpolating polynomial for equally spaced data points are, $x_0 = 0.3, x_1 = 0.55$ and $x_2 = 0.8$ with $h = 0.25$. Consider the quadratic Lagrange interpolating polynomial as

$$f(x) = p_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2), \qquad (1)$$

$$f(0.6) \approx p_2(0.6) = L_0(0.6)(-0.1084) + L_1(0.6)(-0.1808) + L_2(0.6)(-0.1428). \qquad (2)$$

The Lagrange coefficients can be calculate as follows:

$$L_0(0.6) = \frac{(0.6-0.55)(0.6-0.8)}{(0.3-0.55)(0.3-0.8)} = -2/25 = -0.08,$$

$$L_1(0.6) = \frac{(0.6-0.3)(0.6-0.8)}{(0.55-0.3)(0.55-0.8)} = 24/25 = 0.96,$$

$$L_2(0.6) = \frac{(0.6-0.3)(0.6-0.55)}{(0.8-0.3)(0.8-0.55)} = 3/25 = 0.12.$$

Putting these values of the Lagrange coefficients in (2), we have

$$f(0.4) \approx p_2(0.4) = (-0.08)(-0.1084) + (0.96)(-0.1808) + (0.12)(-0.1428) = -0.1821,$$

which is the required approximation of the given exact solution $0.36 \ln 0.6 \approx -0.1839$.
The desired absolute error is

$$|f(0.6) - p_2(0.6)| = |0.36 \ln 0.6 - (-0.1821)| = |-0.1839 + 0.1821| = 0.0018.$$

To compute an error bound for the approximation of the given function in the interval $[0.3, 0.8]$, we use the following quadratic error formula

$$|f(x) - p_2(x)| \leq \frac{Mh^3}{9\sqrt{3}}.$$

As

$$M = \max_{0.3 \leq x \leq 0.8} |f^{(3)}(x)|,$$

and the first three derivatives are

$$f'(x) = 2x \ln x + x, \qquad f''(x) = 2 \ln x + 3, \qquad f^{(3)}(x) = \frac{2}{x},$$

$$M = \max_{0.3 \leq x \leq 0.8} \left| \frac{2}{x} \right| = 20/3 = 6.6667.$$

Hence

$$|f(0.6) - p_2(0.6)| \leq \frac{(6.6667)(0.25)^3}{9\sqrt{(3)}} = 0.0067,$$

which is desired error bound. •

Question 17: Use best integration rule to find the absolute error for the approximation of $\int_0^{1.2} f(x)\, dx$ by using the following set of data points:

x	0.0	0.1	0.21	0.3	0.42	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
$f(x)$	1.0000	1.0950	1.1880	1.2553	1.3331	1.3776	1.4253	1.4648	1.4967	1.5216	1.5403	1.5536	1.5624

The function tabulated is $f(x) = x + \cos x$. How many points approximate the given integral to within accuracy of 10^{-6} ?

Solution. We need only the equally spaced data points, which are as follows

$$x_0 = 0, x_1 = 0.3, x_2 = 0.6, x_3 = 0.9, x_4 = 1.2$$

gives, $h = 0.3$ and $n = (1.2 - 0)/0.3 = 4$, which means the best rule is Simpson's rule. Thus to select the following set of data points for Simpson's rule as

x	0.0	0.3	0.6	0.9	1.2
$f(x)$	1.0000	1.2553	1.4253	1.5216	1.5624

The composite Simpson's rule for five points can be written as

$$\int_0^{1.2} f(x)\, dx \approx S_4(f) = \frac{h}{3}\Big[f(x_0) + 4(f(x_1) + f(x_3)) + 2f(x_2) + f(x_4)\Big],$$

$$\int_0^{1.2} f(x)\, dx \approx 0.1\Big[1.0000 + 4(1.2553 + 1.5216) + 2(1.4253) + 1.5624\Big] = 1.6521.$$

We can easily computed the exact value of the given integral as

$$I(f) = \int_0^{1.2} (x + \cos x)\, dx = (\frac{x^2}{2} + \sin x)\Big|_0^{1.2} = 1.6520.$$

Thus the absolute error $|E|$ in our approximation is given as

$$|E| = |I(f) - S_4(f)| = |1.6520 - 1.6521| = 0.0001.$$

The first four derivatives of the function $f(x) = x + \cos x$ can be obtain as

$$f'(x) = 1 - \sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x.$$

Since $\eta(x)$ is unknown point in $(0, 1.2)$, therefore, the bound $|f^{(4)}|$ on $[0, 1.2]$ is

$$M = \max_{0 \le x \le 1.2} |f^{(4)}| = \max_{0 \le x \le 1.2} |\cos x| = 1.0,$$

at $x = 0$. To find the minimum subintervals for the given accuracy, we use error bound formula of Simpson's rule

$$|E_{S_n}(f)| \le \frac{|-(b-a)^5|}{180n^4} M \le 10^{-6},$$

where $M = \max_{0 \le x \le 1.2} |f^{(4)}(x)| = \max_{0 \le x \le 1.2} |\cos(x)| = 1$, then solving for n, we obtain, $n \ge 10.8432$. Hence to get the required accuracy, we need 12 (even) subintervals which means $n + 1 = 13$ points. •