King Saud University: Mathematics Department Math-254
Second Semester 1445 H Final Examination Solution
Maximum Marks = 40 Time: 180 mins.

Name of the Student:———————— **I.D. No.** ————————

Name of the Teacher:———————— **Section No.** ————————

Note: Check the total number of pages are Six (6).

(**15 Multiple choice questions and Two (2) Full questions**)

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a,b,c,d	a	c	c	b	a	b	c	c	a	a	b	c	a	b	c

Quest. No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 15		30
Q. 16		5
Q. 17		5
Total		40

Question 1: If $f(x) = xe^{-x}$ and $x_0 = 4$, then the first approximation x_1 by Newton's method is:

(a) 5.3333 **(b)** 4.3333 **(c)** 3.3333 **(d)** None of these

Question 2: Using the best iterative formula, the first approximation of the multiple root of the nonlinear equation $1 - \cos x = 0$, taking $x_0 = 0.1$ is:

(a) 0.0980 **(b)** 0.0890 **(c)** 0.0098 **(d)** None of these

Question 3: If the determinant of the Jacobian matrix of the system of nonlinear equations $x^2 + \alpha y^2 = 1,\ xy = 1$ at the point $(1,1)$ is -4, then the value of α is:

(a) 3 **(b)** 2 **(c)** 1.5 **(d)** None of these

Question 4: If the matrix $A = \begin{pmatrix} 3 & 1 \\ 6 & 1 \end{pmatrix}$ is factored as LU using Doolliitle's method, where L is a lower triangular matrix, and U is an upper triangular matrix, then the solution of the system $L\mathbf{y} = [-1,\ 0]^t$ is :

(a) $[-1,\ 2]^t$ **(b)** $[-1,\ -2]^t$ **(c)** $[-1,\ 6]^t$ **(d)** None of these

Question 5: For the linear system $2x+y = 4,\ x+2y = 5$, then the l_∞-norm of the Gauss-Seidal iteration matrix T_G is equal to:

(a) 0.5 **(b)** 0.25 **(c)** 0.75 **(d)** None of these

Question 6: If $A = \begin{pmatrix} 1 & 1 \\ 1 & 0.5 \end{pmatrix}$ and l_∞-norm of the residual vector $\mathbf{r}$ is 0.005, then the error bound for the relative error in the solution of the linear system $A\mathbf{x} = [1, -0.5]^T$ is:

(a) 0.040 **(b)** 0.020 **(c)** 0.025 **(d)** None of these

Question 7: Using data points: $(0, f(0))$ and $(\pi, f(\pi))$. Then the approximation of the function $f(x) = 2\cos x$ at $\pi/2$ by a linear Lagrange polynomial is:

(a) 0.0 **(b)** $\pi/2$ **(c)** $\pi/4$ **(d)** None of these

Question 8: Using data points: $(0,1), (1,2), (2,3)$, if $L_0(1.5) = -0.125$ and $L_2(1.5) = 0.375$, then the approximate value of $f(1.5)$ by a quadratic Lagrange polynomial is:

(a) 2.5 **(b)** 1.5 **(c)** 3.5 **(d)** None of these

Question 9: If $f(x) = \dfrac{3}{x}$ and $f[1,1,1,2] = \alpha$, then α is equal to:

(a) -1.5 **(b)** 1.5 **(c)** -4.5 **(d)** None of these

Question 10: Let $f(x) = x^3$ defined on $[0.2, 0.3]$. Then absolute error using the Two-point difference formula for the approximation of $f'(0.2)$ is:

(a) 0.07 **(b)** 0.12 **(c)** 0.19 **(d)** None of these

Question 11: If $f''(x) = x^4 f(x)$ and $f(0) = 1, f(0.5) = \alpha, f(0.7) = 1.75, f(1) = 2$. Then using 3-point central difference formula for $f''(x)$, the value of α is:

(a) 1.4884 **(b)** 1.9999 **(c)** 1.8484 **(d)** None of these

Question 12: If $f(1) = 0.5, f(1.5) = \alpha, f(1.8) = 1.5, f(2) = 2.5, f(2.5) = 3$ and using the best integration rule the value of $\int_1^{2.5} f(x)\, dx = 3$, then the value of α is:

(a) 1.75 **(b)** 1.5 **(c)** 2.25 **(d)** None of these

Question 13: If $\int_1^2 \dfrac{1}{x+1}\, dx = 0.4055$, then using simple Simpson's rule, the absolute error in the approximation is:

(a) 0.0001 **(b)** 0.0112 **(c)** 0.0025 **(d)** None of these

Question 14: For the initial value problem, $y' + x^2 = y + 1, y(0) = 0.5, n = 1$, if the actual solution of the differential equation is $y(x) = (x+1)^2 - 0.5e^x$, then the absolute error by using Euler's method for the approximation of $y(0.2)$ is:

(a) 0.0293 **(b)** 0.0392 **(c)** 0.0329 **(d)** None of these

Question 15: Given $4y' - y = 0,\ y(0) = 1$, the approximate value of $y(0.5)$ using Taylor's method of order two when $n = 1$ is:

(a) 1.1328 **(b)** 1.1331 **(c)** 1.2839 **(d)** None of these

Question 16: Let $f(x) = \ln(x+2)$ and $x_0 = 0, x_1 = 0, x_2 = 1, x_3 = 1$, find the best approximation of $\ln(2.5)$ by using the cubic Newton's polynomial. Compute absolute error and the error bound.

Solution. Using $f(x) = \ln(x+2)$ and $x_0 = 0, x_1 = 0, x_2 = 1, x_3 = 1$, and $x = 0.5$, the cubic Newton's interpolating polynomial has the following form

$$p_3(x) = f[x_0]+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3].$$

which can be written as

$$p_3(0.5) = f(0)+(0.5-0)f[0,0]+(0.5-0)(0.5-0)f[0,0,1]+(0.5-0)(0.5-0)(0.5-1)f[0,0,1,1],$$

Since $f'(x) = \dfrac{1}{x+2}$, so we find the first, second and third-order divided differences as follows:

$$f[0,0] = \frac{f'(0)}{1!} = f'(0) = \frac{1}{0+2} = 0.5.$$

$$f[0,0,1] = \frac{f[0,1]-f'(0)}{1-0} = f(1)-f(0)-f'(0) = 1.0986-0.6932-0.5 = -0.0946.$$

$$f[0,1,1] = \frac{f[1,1]-f[0,1]}{1-0} = f'(1)-f(1)+f(0) = 0.3333-1.0986+0.6932 = -0.0721,$$

$$f[0,0,1,1] = \frac{f[0,1,1]-f[0,0,1]}{1-0} = -0.0721+0.0946 = 0.0225.$$

$$\ln(2.5) \approx p_3(0.5) = \ln(2)+(0.5)(0.5)+(0.25)(-0.0946)+(-0.1250)(0.0225) = 0.9167,$$

the required approximation of $\ln(2.5)$ and

$$|f(0.5)-p_3(0.5)| = |\ln(2.5)-p_3(0.5)| = |0.9163-0.9167| = 4.0\times 10^{-4},$$

the possible absolute error in the approximation.
Since the error bound for the cubic polynomial $p_3(x)$ is

$$|f(0.5)-p_3(0.5)| = \frac{|f^{(4)}(\eta(x))|}{4!}|(0.5-0)(0.5-0)(0.5-1)(0.5-1)|.$$

Taking the first four derivatives of the given function,

$$f'(x) = \frac{1}{(x+2)}, \quad f''(x) = \frac{-1}{(x+2)^2}, \quad f'''(x) = \frac{2}{(x+2)^3}, \quad f^{(4)}(x) = \frac{-6}{(x+2)^4},$$

and we obtain

$$|f^{(4)}(\eta(x))| = \left|\frac{-6}{(\eta(x)+2)^4}\right|, \quad \text{for} \quad \eta(x)\in(0,1).$$

Since the fourth derivative of the function is decreasing in the interval as

$$|f^{(4)}(0)| = 0.375 \quad \text{and} \quad |f^{(4)}(1)| = 0.0741,$$

so $|f^{(4)}(\eta(x))| \le \max\limits_{0\le x\le 1}\left|\dfrac{-6}{(x+2)^4}\right| = 0.375$ and it gives

$$|f(0.5)-p_3(0.5)| \le \frac{(0.0625)(0.375)}{24} = 9.7656\times 10^{-4},$$

which is the required error bound for the approximation $p_3(0.5)$. •

Question 17: The function tabulated below is $f(x) = x \ln x + x$.

x	0.9000	1.3000	1.5000	1.6000	1.9000	2.3000	2.5000	3.1000
$f(x)$	0.8052	1.6411	2.1082	2.3520	3.1195	4.2157	4.7907	6.6073

Find the approximation of $(\ln 1.9 + 2)$ using three-point formula for $f'(x)$ for smaller value of h. Compute the absolute error and the number of subintervals required to obtain the approximate value of $(\ln 1.9 + 2)$ within the accuracy 10^{-2}.

Solution. Given $f(x) = x \ln x + x$ and $f'(x) = \ln x + 2$, gives $\mathbf{x = 1.9}$. For the given data points we can use all three-points difference formulas with **central difference** at

$$x_0 = 1.5, \qquad \mathbf{x_1 = 1.9}, \qquad x_2 = 2.3, \qquad \text{gives} \quad h = 0.4,$$

for **forward difference** at

$$\mathbf{x_0 = 1.9}, \qquad x_1 = 2.5, \qquad x_2 = 3.1, \qquad \text{gives} \quad h = 0.6,$$

and for **backward difference** at

$$x_0 = 1.3, \qquad x_1 = 1.6, \qquad \mathbf{x_2 = 1.9}, \qquad \text{gives} \quad h = 0.3.$$

So the value of $\mathbf{h = 0.3}$ for the backward difference formula is smaller than both the other formulas. Thus the best three-point formula for the smaller h in this case is the following backward difference formula

$$f'(x_2) \approx \frac{f(x_2 - 2h) - 4f(x_2 - h) + 3f(x_2)}{2h} = D_h f(x_2).$$

Thus using $x_2 = 1.9$ and $h = 0.3$, gives $x_2 - h = 1.6$, and $x_2 - 2h = 1.3$, we have

$$f'(1.9) \approx \frac{f(1.3) - 4f(1.6) + 3f(1.9)}{2(0.3)},$$

$$f'(1.9) \approx \frac{[(1.6411) - 4(2.3520) + 3(3.1195)]}{0.6} = 2.6527.$$

Since the exact value of the derivative $f'(1.9)$ is, 2.6419, therefore, the absolute error $|E|$ can be computed as follows

$$|E| = |f'(1.9) - D_h f(1.9)| = |2.6419 - 2.6527| = 0.0108.$$

The first three derivatives of the given function are as follows

$$f'(x) = \ln x + 2, \qquad f''(x) = \frac{1}{x}, \qquad f''(x) = \frac{-1}{x^2}.$$

Thus

$$M = \max_{1.3 \le x \le 1.9} \left| \frac{-1}{x^2} \right| = \frac{1}{(1.3)^2} = 0.5917.$$

Since the error bound formula of backward difference formula is

$$|E_B(f, h)| \le \frac{h^2}{3} M,$$

and using the given accuracy required 10^{-2}, we have

$$\frac{h^2}{3} M \le 10^{-2}.$$

Then

$$\frac{h^2}{3}(0.5917) \le 10^{-2}, \qquad \text{gives} \qquad h \le \sqrt{\frac{3 \times 10^{-2}}{0.5917}} = 0.2252.$$

Since $\quad n = \dfrac{(1.9 - 1.3)}{0.2252} = 2.6643 \qquad$ and so $\qquad n = 3.$ •