King Saud University: Mathematics Department Math-254
First Semester 1445 H Final Examination Solution
Maximum Marks = 40 Time: 180 mins.

Name of the Student:—————————— **I.D. No.** ——————————

Name of the Teacher:—————————— **Section No.** ——————————

Note: Check the total number of pages are Six (6).

(**15 Multiple choice questions and Two (2) Full questions**)

The Answer Tables for Q.1 to Q.15 : Marks: 2 for each one ($2 \times 15 = 30$)

Ps. : Mark {a, b, c or d} for the correct answer in the box.

Q. No.	1	2	3	4	5	6	7	8	9	10
a,b,c,d	c	b	a	b	c	a	c	b	b	a

Q. No.	11	12	13	14	15
a,b,c,d	c	b	c	b	a

Quest. No.	Marks Obtained	Marks for Questions
Q. 1 to Q. 15		30
Q. 16		5
Q. 17		5
Total		40

Question 1: If $x_{n+1} = \dfrac{a}{b - \cos(x_n)}$, $n \geq 0$, is the fixed-point iterative form of the nonlinear equation $\dfrac{2}{x} + \cos(x) - 3 = 0$, then the value of the constants a and b are:

(a) $a = 2, b = 1$ (b) $a = 3, b = 2$ (c) $a = 2, b = 3$ (d) None of these

Question 2: The next iterative value of the root of $x^3 = 3x - 2$ using the secant method, if the initial guesses are -2.6 and -2.4 is:

(a) -2.2066 (b) -2.1066 (c) -2.3066 (d) None of these

Question 3: If the iterative scheme $x_{n+1} = x_n - k\dfrac{f(x_n)}{f'(x_n)}$, $n \geq 0$, converges at least quadratic to a simple root α, than the value of k is:

(a) k=1 (b) k=2 (c) k=3 (d) None of these

Question 4: The l_∞−norm of the inverse of the Jacobian matrix for the nonlinear system $x^2 + y^2 = 4,\ 2x - y^2 = 0$ using $[x_0, y_0]^t = [1, 1]^t$ is:

(a) 2 (b) 0.5 (c) 4 (d) None of these

Question 5: Let $A = \begin{bmatrix} 1.001 & 1.5 \\ 2 & 3 \end{bmatrix}$, then the determinant of a lower-triangular matrix L of the LU factorization using Crouts method is:

(a) 0.300 (b) 1.001 (c) 0.003 (d) None of these

Question 6: The l_∞−norm of the Jacobi iteration matrix of the following linear system $4x_1 - x_2 + x_3 = 7,\ 4x_1 - 8x_2 + x_3 = -21,\ -2x_1 + x_2 + 5x_3 = 15$ is:

(a) 0.625 (b) 0.5 (c) 0.4 (d) None of these

Question 7: Using Gauss-Seidel method and starting with $\mathbf{x}^{(0)} = [1.200, 0.467, 1.033]^t$, then the first approximation of the solution for the following linear system is:
$5x_1 + 2x_2 - x_3 = 6,\ x_1 + 6x_2 - 3x_3 = 4,\ 2x_1 + x_2 + 4x_3 = 7$ is:

(a) $\mathbf{x}^{(1)} = \begin{pmatrix} 1.024 \\ 1.006 \\ 0.987 \end{pmatrix}$ (b) $\mathbf{x}^{(1)} = \begin{pmatrix} 0.897 \\ 0.950 \\ 1.019 \end{pmatrix}$ (c) $\mathbf{x}^{(1)} = \begin{pmatrix} 1.220 \\ 0.980 \\ 0.895 \end{pmatrix}$ (d) None of these

Question 8: Let $A = \begin{bmatrix} 0 & \alpha \\ 1 & 1 \end{bmatrix}$ and $1 < \alpha < 2$. If the condition number $k(A)$ of the matrix A is 6, then α equals to

(a) 0.8 (b) 0.5 (c) 0.2 (d) None of these

Question 9: Let $x_0 = 2$, $x_1 = 2.5$, $x_2 = 4$ and $x_3 = 5.5$. If the best approximation of $f(x) = \dfrac{1}{x}$ at $x = 3$ using quadratic interpolation formula is $P_2(3) = 0.325$, then the value of the unknown point η in the error formula is equal to :

(a) 2.9201 **(b)** 2.7859 **(c)** 3.1472 **(d)** None of these

Question 10: If $x_0 = 0$, $x_1 = 1$, $x_2 = 3$ and for a function $f(x)$, the divided differences are $f[x_1] = 2$, $f[x_2] = 3$, $f[x_0, x_1] = 1$, $f[x_1, x_2] = \frac{1}{2}$, $f[x_0, x_1, x_2] = -\frac{1}{6}$. Then the approximation of $f(\frac{1}{2})$ using quadratic interpolation Newton formula is:

(a) 1.5417 **(b)** 4.1232 **(c)** 2.3481 **(d)** None of these

Question 11: Let $f(x) = x^3$ and $h = 0.1$. The absolute error for the approximation of $f'(0.2)$ using 2-point forward difference formula is:

(a) 0.0722 **(b)** 0.0711 **(c)** 0.0700 **(d)** None of these

Question 12: The absolute error for the approximation of the integral $\displaystyle\int_1^2 \frac{1}{x+1}\,dx$ using simple Trapezoidal's rule is:

(a) 0.1120 (b) 0.0112 (c) 0.0012 (d) None of these

Question 13: The approximation to the integral $\displaystyle\int_0^2 e^x\,dx$ using simple Simpson's rule is:

(a) 8.4207 (b) 7.4207 (c) 6.4207 (d) None of these

Question 14: For the initial value problem, $(x+1)y' + y^2 = 0, y(0) = 1, n = 1$, if the actual solution of the differential equation is $y(x) = \dfrac{1}{(1 + \ln(x+1))}$, then the absolute error by using Euler's method for the approximation of $y(0.05)$ is:

(a) 0.0350 **(b)** 0.0035 **(c)** 0.0042 **(d)** None of these

Question 15: Using the Taylor's method of order 2 to find the approximate value of $y(0.1)$ for the initial-value problem, $y' = e^{-2x} - 2y,\ y(0) = 0.1, n = 1$, is:

(a) 0.1620 **(b)** 0.1983 **(c)** 0.1846 **(d)** None of these

Question 16: Let $f(x) = \dfrac{3^x}{x}$ and $h = 0.1$ Compute the approximate value of $f''(3)$ and the absolute error. If $\max |f^{(4)}| = 6.1022$, then find the number of subintervals required to obtain the approximate value of $f''(3)$ within the accuracy 10^{-4}.

Solution. Given $x_1 = 3, h = 0.1$, then the formula for $f''(3)$ becomes

$$f''(3) \approx \frac{f(3+0.1) - 2f(3) + f(3-0.1)}{(0.1)^2} = D_h^2 f(3),$$

or

$$f''(3) \approx \frac{f(3.1) - 2f(3) + f(2.9)}{0.01} = \frac{3^{3.1}/3.1 - 2(3^3/3) + 3^{2.9}/2.9}{0.01} \approx 6.2755 = D_h^2 f(3).$$

To compute the absolute error for our approximation we have to compute the second derivative of $f(x)$ as follows:

$$f'(x) = (3^x[\ln(3)x - 1])/x^2,$$

$$f''(x) = (3^x[(\ln(3))^2 x^2 - 2\ln(3)x + 2])/x^3.$$

Since the exact value of $f''(1)$ is

$$f''(3) = (3^3[(\ln(3))^2(3)^2 - 2\ln(3)(3) + 2])/(3)^3 = 6.2709,$$

therefore, the absolute error $|E|$ can be computed as follows:

$$|E| = |f''(1) - D_h^2 f(1)| = |6.2709 - 6.2755| = 0.0046.$$

As the maximum value of the fourth derivative of the given function in the interval is given as

$$M = \max_{2.9 \le x \le 3.1} |f^{(4)}| = 6.1022,$$

and the accuracy required is 10^{-4}, so

$$|E_C(f,h)| \le \frac{h^2}{12} M \le 10^{-4},$$

we have (h=(3.1-2.9)/n)

$$\frac{(3.1-2.9)^2/n^2}{12} M \le 10^{-4}, \quad \text{solving for } n^2, \qquad n^2 \ge \frac{(6.1022)(3.1-2.9)^2(10^4)}{12},$$

by taking square root on both sides, we get, $n \ge 14.2621$, gives, $n = 15$. •

Question 17: Determine the number of subintervals required to approximate the integral $\int_0^2 \frac{1}{x+4} dx$, with an error less than 10^{-4} using composite Simpson's rule. Then approximate the given integral and compute the absolute error.

Solution. We have to use the error formula of composite Simpson's rule which is

$$|E_{S_n}(f)| \leq \frac{(b-a)}{180} h^4 M \leq 10^{-4}.$$

Given the integrand is $f(x) = \frac{1}{x+4}$, and we have $f^{(4)}(x) = \frac{24}{(x+4)^5}$. The maximum value of $|f^{(4)}(x)|$ on the interval $[0,2]$ is $\frac{3}{128}$, and thus $M = \frac{3}{128}$. Using the above error formula and $h = 2/n$, we get

$$\frac{48}{(90 \times 128 \times n^4)} \leq 10^{-4}, \quad \text{or} \quad n^4 \geq \frac{48 \times 10^4}{(90 \times 128)}.$$

Solving for n, gives

$$n \geq \left(\frac{48 \times 10^4}{(90 \times 128)}\right)^{1/4} = 2.5407,$$

so the number of even subintervals n required is $n = 4$. Thus the approximation of the given integral using $h = \frac{2-0}{4} = \frac{1}{2} = 0.5$ is

$$\int_0^2 \frac{1}{x+4} \approx \frac{0.5}{3}\Big[f(0) + 4[f(0.5) + f(1.5)] + 2f(1) + f(2)\Big],$$

$$\int_0^2 \frac{1}{x+4} \approx \frac{1}{6}\Big[0.25 + 4(0.2222 + 0.1818) + 2(0.2) + 0.1667\Big] = 0.4055.$$

Since the exact value of the given integral is

$$\alpha = \int_0^2 \frac{1}{x+4} dx = \ln(1.5) = 0.4055,$$

so the absolute error is

$$AbsE = |exact\ solution - Approximate\ solution| = |0.4055 - 0.4055| = 0.0000,$$

up to 4 decimal places. •