

FINAL EXAMINATION, SEMESTER II, 1445
Department of Mathematics, College of Science
King Saud University
MATH 107 FULL MARK 40 TIME 3 HOURS

Q1. [Marks: 3 + 4 = 7]

(a) Find the value of λ for which the following system of linear equations

$$3x + y + (\lambda^2 - 6)z = \lambda - 3$$

$$2x + 3y + z = -1$$

$$x + 2y + z = 0$$

has: (i) no solution (ii) unique solution (iii) infinitely many solutions.

(b) Use adjoint method to find the *inverse* of the coefficient matrix of the following system, and hence using this *inverse* find the solution of the system

$$x + 2y + 3z = 1$$

$$2x - y = 0$$

$$3x + 3z = 2$$

Q2: [Marks: 2 + 3 + 3 + 3 = 11]

(a) If l has parametric equations $x = 5 - 3t$, $y = -2 + t$, $z = 1 + 9t$, find parametric equations for the line through $P(-6, 4, -3)$ that is parallel to l .

(b) Find an equation of the plane through the point $P(2, 4, -5)$ with normal vector $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

(c) Find the volume of a box having adjacent sides AB , AC , and AD where

$$A(2, 1, -1), B(3, 0, 2), C(4, -2, 1) \text{ and } D(5, -3, 0).$$

(d) Sketch the graph of $4x^2 - 9y^2 + z^2 = 36$ in an xyz -coordinate system, give traces and identify the surface.

Q3: [Marks: 3 + 4 + 3 = 10]

(a) A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial vector $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. If its acceleration $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$, find the velocity and position at time t .

(b) Find tangential component of acceleration and normal component of acceleration for the curve given by $\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}$. Also, find curvature.

(c) Let $f(x, y) = \frac{xy^2}{x^2 + y^4}$. Show that $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.

Q4: [Marks: 3 + 3 + 3 + 3 = 12]

(a) Find the equation of the tangent plane, and parametric equations for the normal line to the graph of $x^3 - 2xy + z^3 + 7y + 6 = 0$ at the point $P(1, 4, -3)$.

(b) Find the directional derivative of $f(x, y, z) = xy + yz + xz$ at the point $A(1, 1, 1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(c) If $g(x, y) = -x^3 + 4xy - 2y^2 + 1$, find the local extrema and saddle points of g .

(d) Let $f(x, y, z) = 2x^2 + y^2 + 3z^2$. Use Lagrange multiplier to find minimum value of f subject to the constraint $2x - 3y - 4z = 49$.

Q, (a)

$$\begin{bmatrix} 3 & 1 & \lambda^2 - 6 & | & \lambda - 3 \\ 2 & 3 & 1 & | & -1 \\ 1 & 2 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & | & 0 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & \lambda^2 - 4 & | & \lambda + 2 \end{bmatrix} \quad (1)$$

- (i) The system has no solution for $\lambda = 2$
since $r_3 \Rightarrow 0z = 4$, so no solution.
- (ii) The system has unique solution for
 $\lambda \neq 2, \lambda \neq -2$
- (iii) The system has infinitely many solutions
for $\lambda = -2$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 3 & 0 & 3 \end{bmatrix}$

Matrix of the cofactor is $\begin{bmatrix} -3 & -6 & 3 \\ -6 & -6 & 6 \\ 3 & 6 & -5 \end{bmatrix} \quad (1)$

$\det(A) = -6$

$\text{Adj}(A) = \begin{bmatrix} -3 & -6 & 3 \\ -6 & -6 & 6 \\ 3 & 6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$A^{-1} = \frac{1}{-6} \begin{bmatrix} 3 & 6 & -3 \\ 6 & 6 & -6 \\ -3 & -6 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 1 & 1 \\ \frac{1}{2} & 1 & -\frac{5}{6} \end{bmatrix} \quad (2)$

$A \underline{x} = B \Rightarrow \underline{x} = A^{-1} B = \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2} \\ -1 & 1 & 1 \\ \frac{1}{2} & 1 & -\frac{5}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{6} \end{bmatrix} \Rightarrow x = -\frac{1}{2}, y = -1, z = \frac{1}{6} \quad (1)$

- Q₂ (a) The direction vector $\underline{a} = \langle -3, 1, 9 \rangle$ for the line l that passes through $P(-6, 4, -3)$. The parametric equations for this line is given by

$$x = -6 - 3t, y = 4 + t, z = -3 + 9t, t \in \mathbb{R}.$$

- (b) The equation of the required plane is $-1(x-2) + 2(y-4) - 3(z+5) = 0$ (2)
 $\Rightarrow x - 2y + 3z + 21 = 0$ (3)

- (c) $\left. \begin{aligned} \underline{a} &= \vec{AB} = \langle 1, -1, 3 \rangle \\ \underline{b} &= \vec{AC} = \langle 2, -3, 2 \rangle \\ \underline{c} &= \vec{AD} = \langle 3, -4, 1 \rangle \end{aligned} \right\}$ (1)

$$\therefore V = |\underline{a} \times \underline{b} \cdot \underline{c}| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & -3 & 2 \\ 3 & -4 & 1 \end{vmatrix} \quad (2)$$

$$\therefore V = 4 \text{ unit of volume.}$$

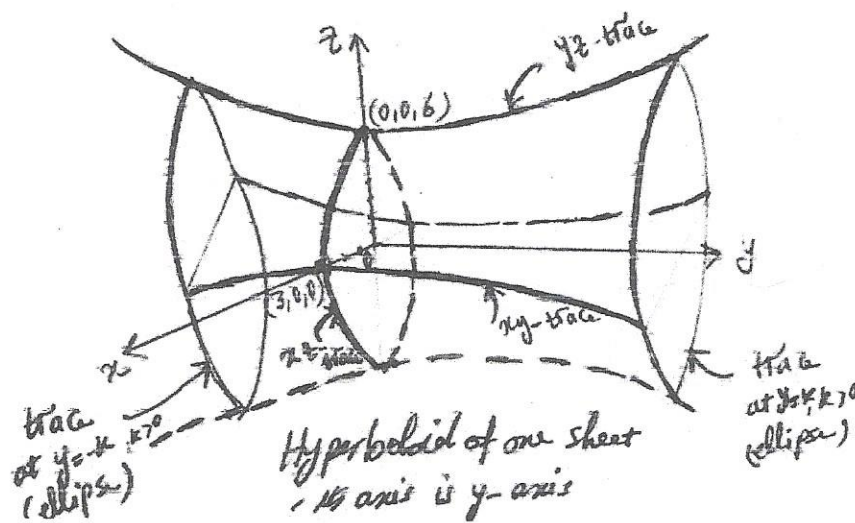
(d) $4x^2 - 9y^2 + z^2 = 36$

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{36} = 1$$

It is hyperboloid of one sheet and its axis is y -axis (1)

[Please See next Page]

Trace	Equation of trace	Description
on xy -plane ($z=0$)	$\frac{x^2}{9} - \frac{y^2}{4} = 1$	Hyperbola
on yz -plane ($x=0$)	$\frac{z^2}{36} - \frac{y^2}{4} = 1$	Hyperbola
on xz -plane ($y=0$)	$\frac{x^2}{9} + \frac{z^2}{36} = 1$	Ellipse



$$Q_3 (a) \quad \underline{r}(0) = \langle 1, 0, 0 \rangle$$

$$\text{Since } \underline{a}(t) = \underline{v}'(t)$$

$$(1.5) \quad \underline{v}(t) = \int \underline{a}(t) dt = \int (4t \underline{i} + 6t \underline{j} + \underline{k}) dt \\ = 2t^2 \underline{i} + 3t^2 \underline{j} + t \underline{k} + \underline{c}$$

$$\text{As } \underline{v}(0) = \underline{c} = \underline{i} + \underline{j} + \underline{k} = \underline{c} \\ \Rightarrow \underline{c} = \underline{i} + \underline{j} + \underline{k} \quad \left. \vphantom{\underline{v}(0)} \right\} \underline{v}(t) = (2t^2+1) \underline{i} + (3t^2+1) \underline{j} + (t+1) \underline{k}$$

$$\text{Since } \underline{v}(t) = \underline{r}'(t)$$

$$\underline{r}(t) = \int \underline{v}(t) dt = \int ((2t^2+1) \underline{i} + (3t^2+1) \underline{j} + (t+1) \underline{k}) dt \\ = \left(\frac{2}{3} t^3 + t \right) \underline{i} + \left(t^3 + t \right) \underline{j} + \left(\frac{t^2}{2} + t \right) \underline{k} + \underline{c}$$

$$\text{As } \underline{r}(0) = \underline{c} \Rightarrow \underline{c} = \underline{i}$$

$$\underline{r}(t) = \left(\frac{2}{3} t^3 + t + 1 \right) \underline{i} + (t^3 + t) \underline{j} + \left(\frac{t^2}{2} + t \right) \underline{k}$$

(1.5)

$$(b) \quad \underline{r}'(t) = 2t \underline{i} + 2t \underline{j} - t^2 \underline{k} \Rightarrow \underline{r}''(t) = 2 \underline{j} - 2t \underline{k}$$

$$\|\underline{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2$$

$$\underline{r}'(t) \cdot \underline{r}''(t) = \langle 2, 2t, -t^2 \rangle \cdot \langle 0, 2, -2t \rangle \\ = 4t + 2t^3 = 2t(t^2 + 2)$$

$$a_T = \frac{\underline{r}'(t) \cdot \underline{r}''(t)}{\|\underline{r}'(t)\|} = \frac{2t(t^2 + 2)}{t^2 + 2} = 2t$$

(1.5)

Q₃

$$(b) \quad \underline{r}'(t) \times \underline{r}''(t)$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 2t & -t \\ 0 & 2 & -2t \end{vmatrix}$$

$$= (-4t^2 + 2t^2)\underline{i} - (-4t - 0)\underline{j} + (4 - 0)\underline{k}$$

$$= -2t^2\underline{i} + 4t\underline{j} + 4\underline{k}$$

$$\|\underline{r}'(t) \times \underline{r}''(t)\|$$

$$= \sqrt{4t^4 + 16t^2 + 16}$$

$$= 2\sqrt{t^4 + 4t^2 + 4}$$

$$= 2(t^2 + 2)$$

$$\therefore a_N = \frac{\|\underline{r}'(t) \times \underline{r}''(t)\|}{\|\underline{r}'(t)\|}$$

$$= \frac{2(t^2 + 2)}{t^2 + 2} = 2$$

1.5

$$K(\text{curvature}) = a_N \frac{1}{\|\underline{r}'(t)\|^2}$$

$$= 2 / (t^2 + 2)^2$$

1

Q3 (c) Take $y = mx$ then

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

$$= \lim_{x \rightarrow 0} f(x, mx)$$

$$= \lim_{x \rightarrow 0} \frac{m^2 x^3}{x^2 + m^4 x^4}$$

(1.5)

$$= \lim_{x \rightarrow 0} \frac{m^2 x}{1 + m^4 x^2} = 0$$

Now use $x = y^2$, then

$$\lim_{(x,y) \rightarrow (0,0)} f(x,y)$$

$$= \lim_{y \rightarrow 0} f(y^2, y)$$

(1.5)

$$= \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

Since different paths lead to different limits, by Two-Path Rule, limit does not exist!

Q4

(a) Let $F = x^3 - 2xy + z^3 + 7y + 6 = 0$

$$\therefore \nabla F = \langle 3x^2 - 2y, -2x + 7, 3z^2 \rangle$$

$$\Rightarrow \nabla F(1, 4, -3) = \langle -5, 5, 27 \rangle$$

Eq. of tangent plane is

$$5x - 5y - 27z = 66$$

(1.5)

The parametric eqs for the normal line are:

$$x = 1 - 5t, y = 4 + 5t, z = -3 + 27t$$

(1.5)

$$Q4 \quad (b) \quad f(x, y, z) = xy + yz + xz$$

$$\nabla f = \langle y+z, x+z, x+y \rangle$$

$$\nabla f(1, 1, 1) = \langle 2, 2, 2 \rangle \quad (1)$$

$$\underline{u} = \frac{\underline{v}}{\|\underline{v}\|} = \frac{\langle 2, 1, -1 \rangle}{\sqrt{4+1+1}} = \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$\underline{D}_{\underline{u}} f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \underline{u} \quad (1)$$

$$= \langle 2, 2, 2 \rangle \cdot \left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$= \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} - \frac{2}{\sqrt{6}}$$

$$= \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3} \quad (1)$$

$$(c) \quad g(x, y) = -x^3 + 4xy - 2y^2 + 1$$

$$g_x = -3x^2 + 4y$$

$$g_{xx} = -6x$$

$$g_y = 4x - 4y$$

$$g_{yy} = -4$$

$$g_{xy} = 4$$

(1)

Q₄ (c) with $g_x = 0$ and $g_y = 0$,

$$-3x^2 + 4y = 0, \quad 4x - 4y = 0$$

$$\Rightarrow y = x = 0, \quad y' = x = \frac{4}{3}$$

$$D = g_{xx}(0,0)g_{yy}(0,0) - [g_{xy}(0,0)]^2 \quad (1)$$

Test, $= 0 - 16 < 0$, by Second Partial
at $(0,0)$ we get saddle point
while at $(\frac{4}{3}, \frac{4}{3})$,

$$D = g_{xx}\left(\frac{4}{3}, \frac{4}{3}\right)g_{yy}\left(\frac{4}{3}, \frac{4}{3}\right) - [g_{xy}\left(\frac{4}{3}, \frac{4}{3}\right)]^2$$

$$= -8(-4) - 16 = 16 > 0 \quad (1)$$

and because $g_{xx}\left(\frac{4}{3}, \frac{4}{3}\right) = -8 < 0$

g has rel. max at $(\frac{4}{3}, \frac{4}{3})$.

(d) Let $g(x,y,z) = 2x - 3y - 4z = 49$
Then, because

$$(1) \quad \nabla f(x,y,z) = 4x\mathbf{i} + 2y\mathbf{j} + 6z\mathbf{k}$$

$$\text{and } \lambda \nabla g(x,y,z) = 2\lambda\mathbf{i} - 3\lambda\mathbf{j} - 4\lambda\mathbf{k}$$

$$(1) \quad \text{We obtain: } \begin{cases} 4x = 2\lambda \\ 2y = -3\lambda \\ 6z = -4\lambda \end{cases} \text{ and } \begin{cases} 2x - 3y - 4z \\ = 49 \end{cases}$$

The solution of the system is $x=3, y=-9, z=-4$.

(1) $\therefore$ the minimum value of f is

$$f(3, -9, -4) = 147$$